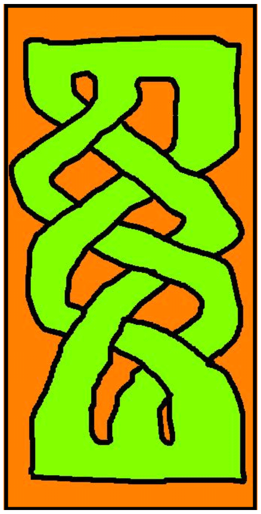

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WHOA! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

November 2018

Puzzler: Here's a curious property of the standard multiplication table.

Choose any top left rectangular portion of it, say the first seven entries of the first three rows.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54

The 21 entries you see have the property that the sum of their cubes equals their sum squared!

$$1^3+2^3+3^3+4^3+5^3+6^3+7^3$$
$$+2^3+4^3+6^3+8^3+10^3+12^3+14^3$$
$$+3^3+6^3+9^3+12^3+15^3+18^3+21^3$$
$$=$$
$$(1+2+3+4+5+6+7+2+4+6+8$$
$$+10+12+14+3+6+9+12+15+18+21)^2$$

The same is true for any rectangular set of entries you choose in the top left corner of the table.

×	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	4	6	8	10	12	14	16
3	3	6	9	12	15	18	21	24
4	4	8	12	16	20	24	28	32
5	5	10	15	20	25	30	35	40

$$1^3+2^3+2^3+4^3+3^3+6^3$$
$$=(1+2+2+4+3+6)^2$$

Moreover, this process is recursive: *Whenever you have two sets of integers satisfying this "the sum of their cubes equals their sum squared" property, use them to make a new multiplication table and generate another set of integers with this property!*

×	1	2	3	4	5	6	7	2	4	6		18	21
1	1	2	3	4	5	6	7	2	4	6		18	21
2	2	4	6	8	10	12	14	4	8	12	…	36	42
2	3	6	9	12	15	18	21	4	8	12		36	42
4	4	8	12	16	20	24	28	8	16	24	…	72	84
3	4	8	12	16	20	24	28	6	12	18		54	63
6	5	10	15	20	25	30	35	12	24	36		108	126

$$1^3+2^3+\cdots+126^3=(1+2+\cdots+126)^2$$

Why do matters work this way?

THE SUM OF THE FIRST N CUBES

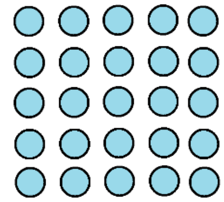
It is well known that the property holds for the top left corner of the multiplication table of one row.

×	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	8	10	12	14

$$1^3+2^3+3^3+\cdots+N^3$$
$$=(1+2+3+\cdots+N)^2$$

I like to demonstrate this by first looking at the diagonals of a square array of dots. There we see the formula

$$1+2+3+\cdots+N+\cdots+3+2+1=N^2.$$



Look at the diagonals and see this sum
$1+2+3+4+5+4+3+2+1$.
This must equal 5^2*, that is, the diagonals account for all twenty-five dots.*

Now look at a square array of numbers in the top-left corner of the multiplication table and ask: *What is the sum of all the entries in that square corner?*

Let's examine this for the five-by-five array.

×	1	2	3	4	5
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
5	5	10	15	20	25

×	1	2	3	4	5
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
5	5	10	15	20	25

Adding by rows gives the sum of entries as

$$(1+2+3+4+5)$$
$$+2\times(1+2+3+4+5)$$
$$+3\times(1+2+3+4+5)$$
$$+4\times(1+2+3+4+5)$$
$$+5\times(1+2+3+4+5)$$

$$=(1+2+3+4+5)\times(1+2+3+4+5)$$

$$=(1+2+3+4+5)^2.$$

Adding by L-shapes gives this sum as

$$(1)+(2+4+2)$$
$$+(3+6+9+6+3)$$
$$+(4+8+12+16+12+8+4)$$
$$+(5+10+15+20+25+20+15+10+5)$$

$$=(1)+2\times(1+2+1)$$
$$+3\times(1+2+3+2+1)$$
$$+4\times(1+2+3+4+3+2+1)$$
$$+5\times(1+2+3+4+5+4+3+2+1)$$

$$=1^3+2^3+3^3+4^3+5^3.$$

Thus

$$(1+2+3+4+5)^2=1^3+2^3+3^3+4^3+5^3$$

and we see that this argument is not specific to the number five.

THE FIRST PART OF THE PUZZLER
Now consider a rectangular top-left region of the multiplication table.

×	1	2	3	4	5	6	…	N
1	1	2	3	4	5	6		N
2	2	4	6	8	10	12		2N
3	3	6	9	12	15	18		3N
⋮								⋮
M	M	2M	3M	4M	5M	6M	⋯	MN

The sum of entries in the region, each cubed, is

$$1^3+2^3+\cdots+N^3$$
$$+2^3+4^3+\cdots+(2N)^3$$
$$\cdots$$
$$+M^3+(2M)^3+\cdots+(MN)^3$$

$$=(1^3+2^3+\cdots+N^3)$$
$$+2^3\times(1^3+2^3+\cdots+N^3)$$
$$\cdots$$
$$+M^3\times(1^3+2^3+\cdots+N^3)$$

$$=(1^3+2^3+\cdots+M^3)(1^3+2^3+\cdots+N^3).$$

By the standard sum of cubes formula, this equals

$$(1+2+\cdots+M)^2(1+2+\cdots+N)^2.$$

But when one multiplies out the product $(1+2+\cdots+M)(1+2+\cdots+N)$ one obtains the sum of all NM entries in the top-left rectangular region of the multiplication table. Thus the sum of entries each cubed actually equals the sum of all entries in the rectangular region squared!

THE RECURSIVE FEATURE

Suppose we have two sets of integer $\{a,b,c,...\}$ and $\{p,q,r,...\}$ satisfying our desired property.

$$a^3+b^3+c^3+\cdots=(a+b+c+\cdots)^2$$
$$p^3+q^3+r^3+\cdots=(p+q+r+\cdots)^2$$

Look at the entries in the multiplication table created by them.

X	a	b	c	...
p	pa	pb	pc	...
q	qa	qb	qc	...
r	ra	rb	rc	...
⋮	⋮	⋮	⋮	

We see all these entries summed together when we expand

$$(a+b+c+\cdots)(p+q+r+\cdots).$$

Thus the sum of entries in this table squared is

$$(a+b+c+\cdots)^2(p+q+r+\cdots)^2$$
$$=(a^3+b^3+c^3+\cdots)(p^3+q^3+r^3+)$$
$$=a^3p^3+a^3q^3+a^3r^3+b^3p^3+\cdots$$

which is the sum of each individual entry cubed!

Everything is hanging together beautifully just as claimed in the opening puzzler.

A CURIOUS NUMBER THEORY CONNECTION

Choose a number, say 36, and list all its factors.

36: 1, 2, 3, 4, 6, 9, 12, 18, 36.

Next count how many factors each factor has.

1 has **1** factor.
2 has **2** factors.
3 has **2** factors.
4 has **3** factors.
6 has **4** factors.
9 has **3** factors.
12 has **6** factors.
18 has **6** factors.
36 has **9** factors.

These counts satisfy

$$1^3+2^3+2^3+3^3+4^3+3^3+6^3+6^3+9^3$$
$$=(1+2+2+3+4+3+6+6+9)^2$$

We've actually proven this curiosity!

Exercise:
Each integer N can be uniquely written as a product of primes. For example, $360=2^3 3^2 5^1$.

a) Show that if $N=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, then N has precisely $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$ factors.

b) Suppose $N=p_1^{\alpha_1}$ is a power of a single prime. List all the factors of N. Count how many factors each factor has. Show that the sum of the cubes of each of these counts is their sum squared.

c) Suppose $N = p_1^{\alpha_1} p_2^{\alpha_2}$ is a product of two prime powers. List all the factors of N. Count how many factors each factor has. Show that the sum of the cubes of each of these counts is their sum squared.

c) Explain why the general number-theory curiosity must hold.

RESEARCH CORNER

The sets of numbers

$$\{1\}$$
$$\{1,2\}$$
$$\{1,2,3\}$$
$$\{1,2,3,4\}$$
$$\vdots$$

each satisfy
$a^3 + b^3 + c^3 + \cdots = (a+b+c+\cdots)^2$.

Our recursive procedure then produces more sets of numbers that satisfy this property too.

Does this recursive procedure produce ALL sets of numbers with this sum-of-cubes property (at least, all positive sets of integers with this property)?

Allowing negative integers into our discussion we see, for example, that $\{-3,-1,\ 1,\ 3\}$ is a set with the desired property, although somewhat trivially. Are there interesting non-trivial examples of sets with negative entries?

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