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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

NOVEMBER 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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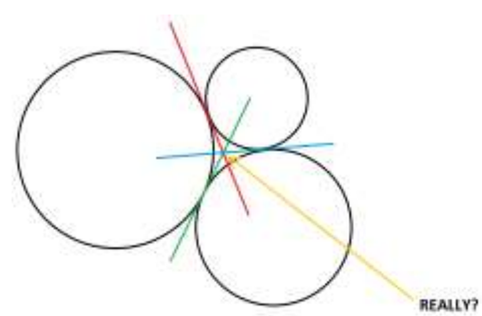
The MAA's Curriculum Inspirations project is still going strong: www.maa.org/ci. See lots of strategies for solving problems and for teaching problem solving. Videos and written essays galore of worked examples. What a resource!

OPENING PUZZLE: Three circles are pairwise tangent. Are the three pairwise common tangent lines sure to meet at a common point?



ALGEBRA IN GEOMETRY

Most every graph or curve one draws on a coordinate plane in high-school is given by an equation of the form $h(x,y)=0$. For example, a circle with center the origin and radius 1 is given by:

$$h(x,y)=x^2+y^2-1=0.$$

The absolute value graph is given by:

$$h(x,y)=y-|x|=0,$$

and, in fact, the graph of any function $y=f(x)$ is given by:

$$h(x,y)=y-f(x)=0.$$

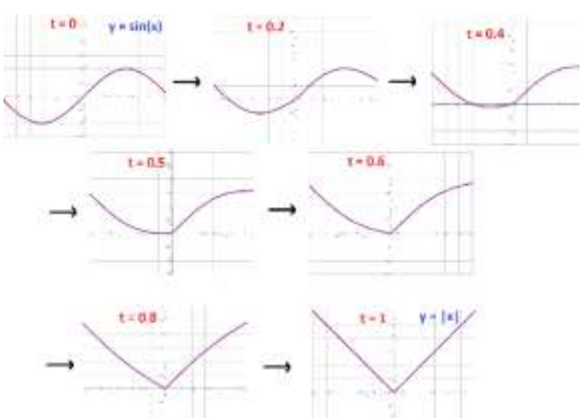
And if we have two curves:

$$h_1(x,y)=0$$
$$h_2(x,y)=0$$

then the equation

$$(1-t)\cdot h_1(x,y)+t\cdot h_2(x,y)=0$$

gives a whole host of new curves, one for each value of t. For $t=0$, we get the first curve, $h_1(x,y)=0$. For $t=1$, we get the second curve, $h_2(x,y)=0$. For intermediate "times" t we get intermediate curves. Thus if we let t vary from $t=0$ to $t=1$ we see the first curve smoothly morph to become the second curve. Here's a series showing the graph of $y=\sin(x)$ transforming into the graph of $y=|x|$.



Pushing the Algebra Further:

Given two curves $h_1(x,y)=0$ and $h_2(x,y)=0$, what can we say, in general, about the curve

$$a\cdot h_1(x,y)+b\cdot h_2(x,y)=0$$

for a pair of nonzero real values a and b?

For starters, if (p,q) is a point that lies on both curves (and so $h_1(p,q)=0$ and $h_2(p,q)=0$), then (p,q) lies on the curve $a\cdot h_1(x,y)+b\cdot h_2(x,y)=0$ as well. Thus:

> *The curve $a\cdot h_1(x,y)+b\cdot h_2(x,y)=0$ passes through all intersection points of the original two curves.*

Suppose, instead, (p,q) lies on one curve, say the first, but not the second. (So $h_1(p,q)=0$, but $h_2(p,q)\neq 0$.) Then $a\cdot h_1(p,q)+b\cdot h_2(p,q)\neq 0$.

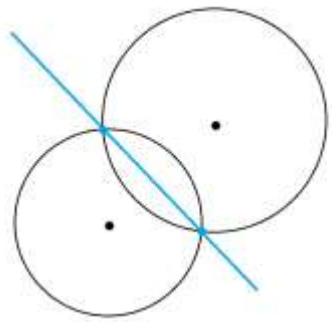
> *The curve $a\cdot h_1(x,y)+b\cdot h_2(x,y)=0$ only intersects the original curves at intersection points of the two curves.*

Consequently

> *If the curves $h_1(x,y)=0$ and $h_2(x,y)=0$ do not intersect, then the curve $a\cdot h_1(x,y)+b\cdot h_2(x,y)=0$ does not intersect either of the original curves.*

Let's have some fun with these ideas.

Puzzle 1: Find the equation of the line that passes through the two intersection points of the circles $(x-1)^2 + y^2 = 4$ and $(x-3)^2 + (y-4)^2 = 10$.



Answer: Each of the curves

$$a\left((x-1)^2 + y^2 - 4\right) + b\left((x-3)^2 + (y-4)^2 - 10\right) = 0$$

passes through the two intersection points of the two circles. Expanding, we have

$$(a+b)x^2 + (a+b)y^2 - (2a+6b)x - 8by - 3a + 15b = 0.$$

If $a+b \neq 0$, this is the equation of a circle. (Thus we have found equations for infinitely many different circles that pass through the two intersection points, one for each pair of real values a, b with $a+b \neq 0$.)

But choose instead $a=1$ and $b=-1$, and we get an equation with no squared terms. We get

$$4x + 8y - 18 = 0.$$

This is the equation of a (in fact, the!) straight line that passes through the two intersection points of the circles.

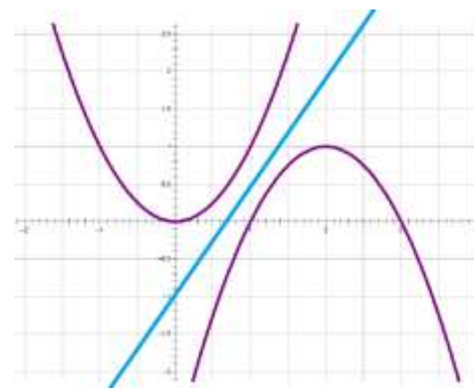
SUMMARY: To find the equation of the line between connecting the intersection points of two intersecting circles, just subtract the equations of the two circles.

Circle 1: $(x-a)^2 + (y-b)^2 = r^2$

Circle 2: $(x-c)^2 + (y-d)^2 = s^2$

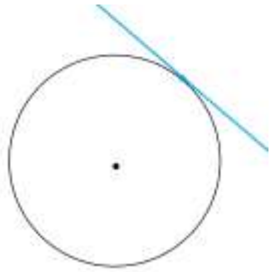
The difference is a linear equation which must be the equation of the desired line.

Puzzle 2: Find the equation of a line that separates the parabolas $y = x^2$ and $y = -x^2 + 4x - 3$.



Answer: Since the two original parabolas do not intersect, none of the curves $a\left(y - x^2\right) + b\left(y + x^2 - 4x + 3\right) = 0$ intersect with either parabola. We can see that one of these equations is a straight line if we choose values for a and b that have the x^2 terms cancel: $a = 1$, $b = 1$ will do. Thus $2y - 4x + 3 = 0$ is the equation of the line that sits between the two parabolas.

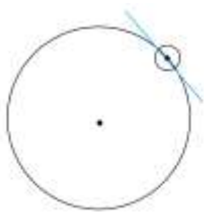
Puzzle 3: Find the equation of the line tangent line to the circle $x^2 + y^2 = r^2$ at the point (p,q) on the circle.



Answer: Write the equation of the circle as $x \cdot x + y \cdot y = r^2$. Then I claim that the equation of the tangent line at the point (p,q) is $px + qy = r^2$.

Consider a second, tiny circle, with center (p,q) and small radius h:

$$(x-p)^2 + (y-q)^2 = h^2.$$



The line passing through the two intersection points of the circles approximates the tangent line. (And the smaller h is, the better the approximation.) Following the solution to puzzle 1, this line has equation:

$$1 \cdot \left(x^2 + y^2 - r^2\right) + (-1) \cdot \left((x-p)^2 + (y-q)^2 - h^2\right) = 0$$

that is,

$$2px + 2qy = r^2 + p^2 + q^2 - h^2.$$

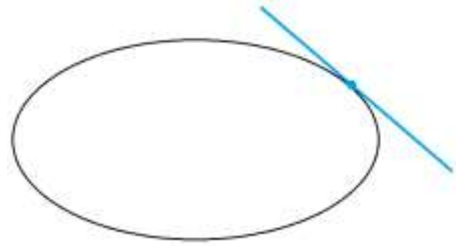
But the point (p,q) is on the circle, and so $p^2 + q^2 = r^2$ and our equation is equivalent to $px + qy = r^2 - \frac{h^2}{2}$.

This approaches the equation

$$px + qy = r^2$$

as h gets smaller and smaller. This must be the equation of the tangent line.

Puzzle 4: Show the equation of the line tangent line to the ellipse $\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1$ at the point (p,q) on the ellipse is $\frac{px}{m^2} + \frac{qy}{n^2} = 1$.



Answer: Consider the small ellipse $\frac{(x-p)^2}{m^2} + \frac{(y-q)^2}{n^2} = h^2$ centered about (p,q) with small "radius" h. The line through the two intersection points,

$$\left(\frac{x^2}{m^2} + \frac{y^2}{n^2} - 1\right) - \left(\frac{(x-p)^2}{m^2} + \frac{(y-q)^2}{n^2} - h^2\right) = 0,$$

that is,

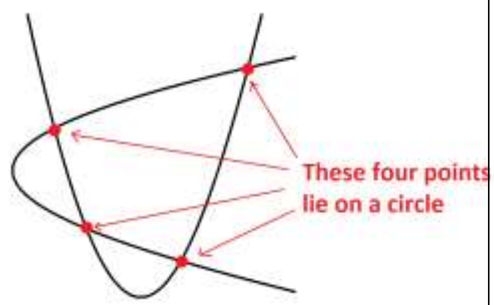
$$\frac{2px}{m^2} + \frac{2qy}{n^2} = 1 + \frac{p^2}{m^2} + \frac{q^2}{n^2} - h^2 = 2 - h^2,$$

approximates the tangent line. As $h \to 0$, this becomes the equation

$$\frac{px}{m^2} + \frac{qy}{n^2} = 1.$$

Extra: How could you find the equation of tangent line common to two touching parabolas?

Puzzle 5: Show that if two "orthogonal" parabolas $y = x^2 + bx + c$ and $x = y^2 + ey + f$ intersect in four points, then those four points lie on a circle.

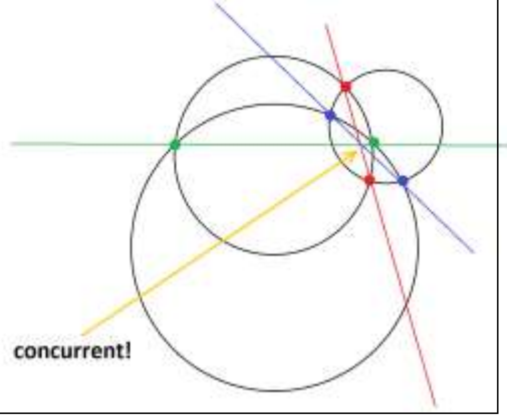


Answer: The equation

$$1 \cdot \left(x^2 + bx + c - y\right) + 1 \cdot \left(y^2 + ey + f - x\right) = 0$$

is the equation of a circle that contains all four intersection points.

Puzzle 7: Three circles intersect pairwise. For each pair of intersection points for two circles, draw the line through those two points. Show that if the three lines so constructed are not parallel, then they pass through a common point.



Towards the Answer: Suppose the three circles have the equations $h_1(x,y) = 0$, $h_2(x,y) = 0$, and $h_3(x,y) = 0$. Then the equations of the lines are:

$$h_1(x,y) - h_2(x,y) = 0,$$

$$h_2(x,y) - h_3(x,y) = 0,$$

and $$h_3(x,y) - h_1(x,y) = 0.$$

If (p,q) is the point of intersection of the first two lines, then why does it follow that this point lies on the third line as well?

Extra: Is it possible for three pairwise intersecting circles to produce three parallel intersection-point lines? Precisely two parallel intersection-point lines?

RESEARCH CORNER

Is it possible to solve the opening puzzler in the algebraic style of puzzle 7?

Is it possible to adapt the methods of this essay to find the tangent planes to spheres and ellipsoids?

Is it possible to find the equation of the plane that contains the circle of two intersecting spheres?

If four spheres intersect pairwise, do the six planes containing the circles of intersection meet at a common point or along a common line?

In what ways can the puzzles in this essay be extended to higher dimensions?

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