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WOCKIN' COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

NOVEMBER 2014

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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Have a look at Christopher Walker's "Sine Rider" game. It is pretty cool! http://sineridergame.com

This month, let's have some polynomial fun. Here are ten cool puzzles all to do with polynomials.

To be clear on the language and the jargon… A function p is deemed a *polynomial* function if, possibly after some algebraic manipulation, it can be written in the form:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0$$

for some real numbers a_n, a_{n-1}, …, a_1, a_0. (called the *coefficients* of the polynomial) and some non-negative integer n (called the *degree* of the polynomial).

1: Deducing a polynomial
I am told that a particular polynomial p has non-negative integer coefficients and I am charged with the task of deducing what the polynomial is. But to complete this task I am allowed to ask only two questions about the polynomial and each question must be of the form: "What output value does the polynomial give if we provide in an input of __?"

Do I have any hope of successfully deducing full knowledge of the polynomial?

2: Deducing again
I am told that each coefficient of a secret polynomial p is either $-1, 0$, or 1. Given that $p(3) = 65$, what is the polynomial?

3: Counting polynomials
How many polynomials p with non-negative integer coefficients are there with $p(10) = 111$?

4: Consecutive outputs and inputs
A polynomial p with integer coefficients has $p(a) = 99$, $p(b) = 100$, and $p(c) = 101$ for three integer inputs a, b, and c. Explain why a, b, and c must be consecutive integers.

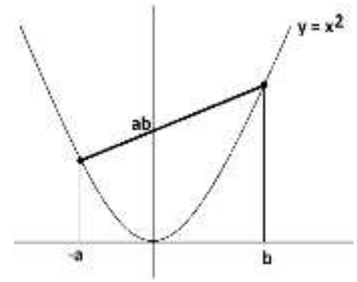
5: Input/output switch
For any two distinct integers a and b there is a certainly a polynomial p with $p(a) = b$ and $p(b) = a$. (Take p to be the linear function through (a,b) and (b,a), for example: $p(x) = a + b - x$.)

Give an example of three distinct integers a, b, and c and a polynomial p with integer coefficients satisfying $p(a) = b$, $p(b) = c$, and $p(c) = a$.

6: Polynomial identity crisis
A polynomial p of degree $n-1$ very much wants to be the function $f(x) = \frac{1}{x}$. It satisfies $p(1) = 1$, $p(2) = \frac{1}{2}$, $p(3) = \frac{1}{3}$, all the way up to $p(n) = \frac{1}{n}$. What is the value of $p(n+1)$?

7: Always a multiple of three
A polynomial p with integer coefficients has $p(-1)$, $p(0)$, and $p(1)$ each a multiple of three. Prove that $p(n)$ is also sure to be a multiple of three for each integer n.

8: Parabolic products
We saw in last month's COOL MATH essay that the parabola $p(x) = x^2$ has the property that, for any two real numbers a and b, the line connecting the points $(-a, p(-a))$ and $(b, p(b))$ on the graph of the curve crosses the y-axis at the product ab.



Finding products with a parabola.

It is irritating that we must work with "$-a$" rather than a directly.

Is there a function f with the property that, for any two real numbers a and b, the line through the points $(a, f(a))$ and $(b, f(b))$ on the graph of the function crosses the y-axis at the product ab?

9: Cubic relations
Where does the line connecting the points (a,a^3) and (b,b^3), for two real numbers a and b, again intercept the curve $y=x^3$?

10: Parabolic variations
In mathematics $\lfloor x \rfloor$ denotes the "floor" of the real number x. It is the largest integer less than or equal to x. (So, for example $\lfloor 1.99 \rfloor = 1$, $\lfloor \pi \rfloor = 3$, and $\lfloor -10.87 \rfloor = -11$.)

The graph of the curve $y=x^2$ is a parabola through the origin.

Can you predict what the graphs of each of the following relations look like?

$$y=\left(\lfloor x \rfloor\right)^2$$
$$y=x\lfloor x \rfloor$$
$$\lfloor y \rfloor=\left(\lfloor x \rfloor\right)^2$$

SOME KEY POLYNOMIAL MATHEMATICS
Polynomials are very similar in their structure to the structure of the integers. Each integer in base ten is an expression of the form:

$$a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0$$

and each polynomial is an analogous expression given in "base x." We usually restrict the coefficients a_n, a_{n-1}, …, a_0 in base ten arithmetic to single digit integers. But we don't have to! We could write $2\,|\,12\,|\,11$ ("two hundred and twelvety eleventy") for the number $2\times100+12\times10+11=331$ if we wished, or $3.7\,|\,-1$ for the number 36. Polynomials usually have no restrictions on the types of numbers we use for their coefficients. Several of the puzzles in the list make use of this interplay between polynomials and base-ten, or other base, expressions for integers.

If

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0$$

then

$$p(x)-p(r) = a_n\left(x^n-r^n\right)+\cdots+a_2\left(x^2-r^2\right)+a_1\left(x-r\right)$$

Since x^k-r^k is always divisible by $x-r$:

$$x^k-y^k=(x-y)\left(x^{k-1}+x^{k-2}y+\cdots+y^{k-1}\right)$$

we see that $x-r$ is always a factor of $p(x)-p(r)$ for any polynomial p. Some puzzles in this essay use this observation.

There are some nice consequences too of this observation:

If $p(r)=0$, that is, r is a root of the polynomial, then we have that $x-r$ is a factor of $p(x)-p(r)=p(x)$. (I guess this is called the *Factor Theorem*.)

The quadratic formula shows that any quadratic has at least one (possibly complex) root. It follows then that every quadratic can be written in the form: $p(x)=a_2(x-r_1)(x-r_2)$.

Every real cubic graph crosses the x-axis at least once and so has at least one real root. This means, along with the fact that every quadratic factors, that every cubic factors too: $p(x)=a_3(x-r_1)(x-r_2)(x-r_3)$.

(Matters are far less direct for higher-degree polynomials!)

PUZZLE ANSWERS

1. First ask for the value of $p(1)$ and call this value $b-1$. This means that if $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots a_2x^2+a_1x+a_0$, then $a_n+a_{n-1}+\cdots+a_1+a_0=b-1$. Since each coefficient is a non-negative integer, we have that $0\le a_i\le b-1$ for each i.

Now ask for the value of $p(b)$. Call this number N. We have that $N=a_nb^n+a_{n-1}b^{n-1}+\cdots+a_1b+a_0$. So all you need do is write N in base b and read off the coefficients a_i! (For example, the only polynomial p with non-negative integer coefficients satisfying $p(1)=9$ and $p(10)=1062$ is $p(x)=x^3+6x+2$.)

2. Every integer has a unique representation in base three using the digits 0, 1, and 2. For example,

$$\begin{aligned}65&=2\times3^3+1\times3^2+0\times3+2\times1\\&=2\,|\,1\,|\,0\,|\,2\end{aligned}$$

As

$$2\times3^k=1\times3^{k+1}+(-1)\times3^k$$

any digit of 2 can be replaced with a "digit" of -1 with the addition of a 1 to the digit to its left:

$$\begin{aligned}65&=2\,|\,1\,|\,0\,|\,2\\&=2\,|\,1\,|\,1\,|\,-1\\&=1\,|\,-1\,|\,1\,|\,1\,|\,-1\end{aligned}$$

(There may be "carries" involved, for example: $2\,|\,2\ =\ 3\,|\,-1\ =\ 1\,|\,0\,|\,-1$.)

We see that every number can be written in base three using the digits -1, 0, and 1. Also, such a representation is unique. (If $N=a_n3^n+\cdots+a_13+a_0$ and $N=b_m3^m+\cdots+b_13+b_0$, then a_0 is its remainder upon division by three, as is b_0. Thus $a_o=b_0$. It then follows that $a_n3^{n-1}+\cdots a_23+a_1$ and $b_n3^{n-1}+\cdots b_23+b_1$ represent the same integer, and so $a_1=b_1$, by the same reasoning. And so on.)

Now we are told
$p(3)=a_n3^n+\cdots+a_13+a_0=65$
with each coefficient equal to -1, 0, or 1. As there is only one way to write 65 as such a sum of powers of three, we must have $p(x)=x^4-x^3+x^2+x-1$.

3. Now after the first two questions we can't help but think base arithmetic. If
$p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots a_2x^2+a_1x+a_0$
then $p(10)$ is the base-ten number:

$$a_n\,|\,a_{n-1}\,|\cdots|\,a_2\,|\,a_1\,|\,a_0\ .$$

Each a_i is a non-negative integer, but with no bound on its size. So we must consider all the ways to write 111 in base ten without regard to the size of the digits we use. There are 13 ways to do this.

$$\begin{aligned}111&=1\,|\,1\,|\,1\\&=0\,|\,11\,|\,1=0\,|\,10\,|\,11=0\,|\,9\,|\,21=\cdots0\,|\,1\,|\,101\\&=0\,|\,0\,|\,111\end{aligned}$$

These correspond to 13 polynomials p, with non-negative integer coefficients, satisfying $p(10)=111$, namely,
$p(x)=x^2+x+1$, $p(x)=11x+1$, $p(x)=10x+11$, $p(x)=9x+21$, …, $p(x)=x+101$, and $p(x)=111$.

4. We have that $x-r$ is a factor of $p(x)-p(r)$ for any polynomial p.

So if $p(b)=100$ and $p(a)=99$, then $b-a$ is a factor of $p(b)-p(a)=1$, giving that $b-a=\pm1$. We have that a and b are consecutive integers. Similarly, b and c are consecutive integers. Since

$a \neq c$, it must be that all three integers a, b, and c are consecutive.

5. Suppose such a polynomial and three integers exist. Let's assume a is the smallest of the three integers and c is the largest. Now $c-a$ is a factor of $p(c)-p(a)=a-b$. But $c-a$ is a larger number than $b-a$, so this is impossible! One cannot find a polynomial and three integers with these properties!

6. We have that $q(x)=xp(x)-1$ is a degree n polynomial with $q(1)=0$, $q(2)=0$, up to $q(n)=0$. By the Factor Theorem it follows that $q(x)=a(x-1)(x-2)\cdots(x-n)$ for some real number a. So we have:

$$p(x)=\frac{a(x-1)(x-2)\cdots(x-n)+1}{x}.$$

For this to be a polynomial it better be the case that the numerator has no constant term. This means we must have $a(-1)(-2)\cdots(-n)=-1$, giving $a=(-1)^{n+1}/n!$. It follows that $p(n+1)=\frac{(-1)^{n+1}+1}{n+1}$, which is $\frac{0}{n+1}$ if n is even, $\frac{2}{n+1}$ if n is odd (and, alas, it is not $\frac{1}{n+1}$.)

Question: What is $p(0)$?

7. We have that, for any k, $p(3k)-p(0)$ is a multiple of $3k-0=3k$. This means that:

$$p(3k)=3k(something)+p(0).$$

Also $p(3k+1)-p(1)$ is a multiple of $(3k+1)-1=3k$. So:

$$p(3k+1)=3k(something)+p(1).$$

Also, $p(3k-1)-p(-1)$ is a multiple of $3k$ giving:

$$p(3k-1)=3k(somthing)+p(-1).$$

As each of $p(-1)$, $p(0)$, and $p(1)$ is a multiple of three, it follows that each of $p(3k)$, $p(3k+1)$, and $p(3k-1)$ is a multiple of three for each k. Thus $p(n)$ is a multiple of three for any n.

8. We seek a function f so that the points $(a, f(a))$, $(b, f(b))$, and $(0, ab)$ are sure to be collinear. We need matching slopes:

$$\frac{f(a)-ab}{a}=\frac{f(b)-ab}{b}.$$

This gives:

$$f(b)=-b^2-b(something)$$

suggesting that a function of the form

$$f(x)=kx-x^2$$

for some value k might work. One checks that it does – for any value of k you like. (And note … parabolas again!)

9. Let $L(x)$ be the linear function with $L(a)=a^3$ and $L(b)=b^3$. (So $L(x)=mx+c$ for some values m and c.) Then $p(x)=x^3-L(x)$ is a cubic with $p(a)=0$ and $p(b)=0$. By the Factor Theorem we have:

$$p(x)=k(x-a)(x-b)(x-r)$$

for some value k and some third root r.

Notice that $p(r)=0$, giving $L(r)=r^3$. Thus $x=r$ is the location at which the line crosses the curve $y=x^3$ a third time.

Now $p(x)=x^3-L(x)=x^3-mx-c$.

But also, $p(x)=k(x-a)(x-b)(x-r)$.

We must have $k=1$, and

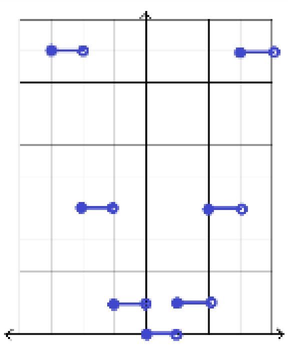
$$\begin{aligned} p(x) &= (x-a)(x-b)(x-r) \\ &= x^3-(a+b+r)x^2 \\ &\quad +(ab+ar+br)x-abr \end{aligned}$$

showing that $a+b+r=0$, that is, $r=-(a+b)$.

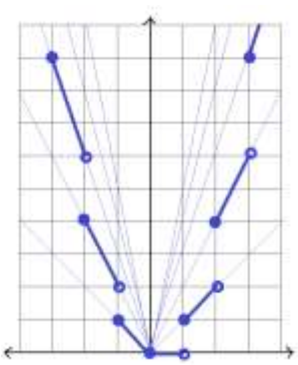
The line that crosses the graph of $y=x^3$ at (a,a^3) and (b,b^3) does so a third time at $\left(-(a+b),-(a+b)^3\right)$.

10. We have:

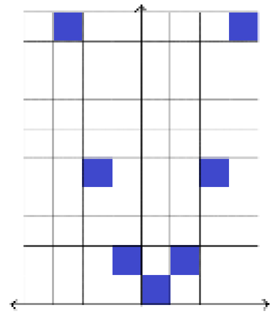
$y=\lfloor x\rfloor^2$



$y=x\lfloor x\rfloor$



$\lfloor y\rfloor=\lfloor x\rfloor^2$



Question: I am not being careful with my "endpoints" in this third graph. What is the accurate description of these boundary points?

RESEARCH CORNER:

It is fun to play around with chunky versions of various graphs. For example, take the equation of a circle $x^2+y^2=1$ or a hyperbola $x^2-y^2=1$ and play with their variations using the floor function $\lfloor x\rfloor$, the ceiling function $\lceil x\rceil$ (the smallest integer greater than or equal to x), and/or the fractional part function $\{x\}=x-\lfloor x\rfloor$. For example, can you predict the graphs of these "circles"?

$$x\{x\}+y\{y\}=1$$
$$\{x\}^2+\{y\}^2=1$$

A more general question: Given a function $y=f(x)$ and its graph, is there anything meaningful and general that can be said about the graphs of $\{y\}=f(\{x\})$ or $y=f(\lceil x\rceil)$ and so on?

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