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GLOBAL MATH PROJECT

Uplifting Mathematics for All

WHOA! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MAY 2021

THIS MONTHS' PUZZLER:

We have

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

Observe

$$P_1(-n)=\frac{(-n)(-n+1)}{2}$$
$$=\frac{(n-1)n}{2}=P_1(n-1)$$

$$P_2(-n)=\frac{(-n)(-n+1)(-2n+1)}{6}$$
$$=-P_2(n-1)$$

and

$$P_3(-n) = \frac{(-n)^2(-n-1)^2}{4} = P_3(n-1).$$

In the video featured in this essay I establish general formulas for the sums of powers.

$$P_k(n) = 1^k + 2^k + 3^k + \cdots + n^k.$$

Is is always true that

$$P_k(-n) = P_k(n-1) \text{ if } k \text{ is odd}$$

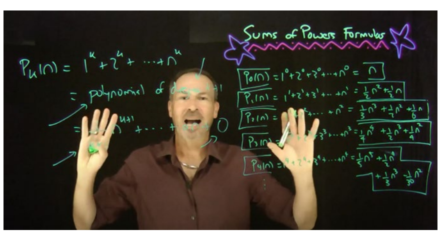
and

$$P_k(-n) = -P_k(n-1) \text{ if } k \text{ is even?}$$

Comment: Another way of saying this is that $P_k\left(-x-\frac{1}{2}\right) = (-1)^{k+1} P_k\left(x-\frac{1}{2}\right)$.

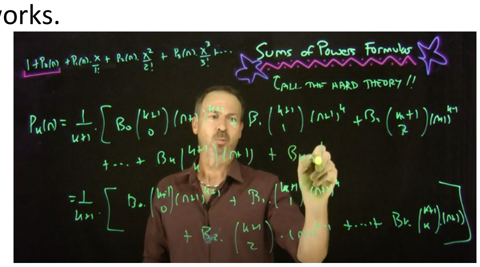
SUMS OF POWERS

I recently had some fun making two videos deriving the sums of powers of formulas, not skimping on any of the math. I used the properties of the famous arithmetic triangle to get there.



https://youtu.be/-rGJc8aLWZU

I ended the first video with a wild and whacky pseudo-calculus approach to crank out these formulas with absurd ease. This second—challenging—video explains why it works.



https://youtu.be/IAMfTu5bzg4

But in these videos I did not go through some more straightforward ways to get to the first three formulas:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

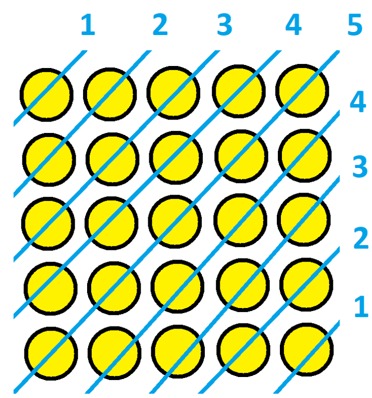
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

For completeness of the experience, I thought I would go through some approaches here.

The sum of the first n counting numbers

This picture shows that $1+2+3+4+5+4+3+2+1 = 5^2$.



Look at the diagonals of an n-by-n square and we'll see $1 + 2 + \cdots + (n-1) + n + (n-1) + \cdots + 2 + 1 = n^2$

Add n to each side and get

$$2(1 + 2 + \cdots + n) = n^2 + n$$

which gives the formula for $P_1(n)$.

The sum of the first n cube numbers

What is the sum of all the products in a five-by-five multiplication table?

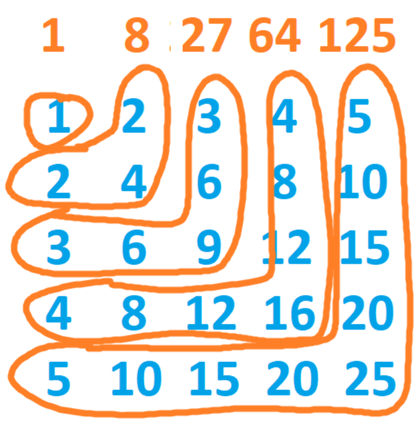
x	1	2	3	4	5
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
5	5	10	15	20	25

The sum the entries in the k th row is $k(1+2+3+4+5)$ and so the sum of entries in all five rows is

$$(1+2+3+4+5)(1+2+3+4+5),$$

which equals $\left(\frac{5^2+5}{2}\right)^2$.

Now look at gnomons of entries in the table (L-shapes). Those entries sum to cube numbers!



For example, the fourth gnomons has entries summing to

$$1\times4+2\times4+3\times4+4\times4+4\times3+4\times2+4\times1$$
$$=4(1+2+3+4+3+2+1)$$
$$=4\times4^2$$
$$=4^3$$

Thus, the sum of all entries in the table is the sum of these cube numbers. Thus $1^3+2^3+3^3+4^3+5^3$ must equal $\left(\frac{5^2+5}{2}\right)^2$.

In general, $P_3(n)=\left(\frac{n^2+n}{2}\right)^2$.

The sum of the first n square numbers

The sum $1^2+2^2+3^2+4^2+5^2$ equals the sum shown.

$$\begin{aligned}&1\\+\,&1+2+1\\+\,&1+2+3+2+1\\+\,&1+2+3+4+3+2+1\\+\,&1+2+3+4+5+4+3+2+1\end{aligned}$$

Thus, the sum of the first five square numbers is the sum of all the entries in these two triangular rays of entries.

1								
1	2				1			
1	2	3			1	2		
1	2	3	4		1	2	3	
1	2	3	4	5	1	2	3	4

Looking column-wise, we see the entries in the blue array sum to

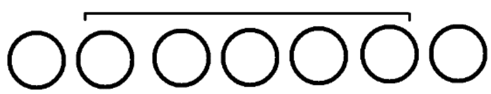
$$5\times1+4\times2+3\times3+2\times4+1\times5$$

and in the purple array to

$$4\times1+3\times2+2\times3+1\times4.$$

Let's first give meaning to the sum $5\times1+4\times2+3\times3+2\times4+1\times5$.

Consider this question: *Given a row of* 7 *dots, in how many ways can we color three of the dots blue?*

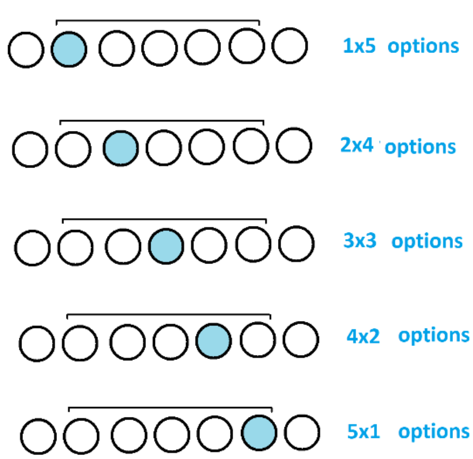


Answer 1: There are $\frac{7!}{3!4!}$ to select three dots to be colored blue.

Answer 2: For any three dots colored, the middle one must be among the central five dots. So, let's now ask: For each possible location of the middle blue dot, how many choices are there for the placement of the left blue dot and for the placement of the blue right dot? The sum of all these options gives the total count we seek.

And that count is

$$1\times5+2\times4+3\times3+4\times4+5\times1!$$



We have that

$$1\times5+2\times4+3\times3+4\times2+5\times1$$

equals $\frac{7!}{3!4!}$.

In general, counting how many ways we can color three dots blue from a row of $n+2$ dots two different ways establishes

$$1\times n+2\times(n-1)+\cdots+n\times1 = \frac{(n+2)(n+1)n}{6}$$

We can now compute

$$1^2+2^2+3^2+4^2+5^2 .$$

It equals

$$5\times1+4\times2+3\times3+2\times4+1\times5=\frac{7!}{3!4!}$$

plus

$$4\times1+3\times2+2\times3+1\times4=\frac{6!}{3!3!}$$

and so has value

$$\frac{7\times6\times5}{3!}+\frac{6\times5\times4}{3!}=\frac{5\times6\times(7+4)}{3!}=55$$

In general, $P_2(n)=1^2+2^2+\cdots+n^2$ has value

$$\frac{(n+2)(n+1)n}{3!}+\frac{(n+1)n(n-1)}{3!} = \frac{n(n+1)(n+2+n-1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

RESEARCH CORNER

1. There are a myriad of clever (many purely visual) proofs of various summation formulas. Explore the internet for more. Can you create a brand new proof of one yourself?

2. Plot in the complex plane the roots of each summation polynomial. Any structure of note?

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