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GLOBAL MATH PROJECT

Uplifting Mathematics for All

WILD COOL MATH!

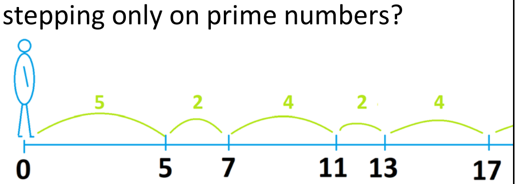
CURIOUS MATHEMATICS FOR FUN AND JOY

MARCH 2021

THIS MONTHS' PUZZLER:

Start at position 0 on the number line.

a) Walking steps of sizes 1, 2, 3, 4, or 5, can you walk infinetly far to the right stepping only on prime numbers?



b) If no, how about taking steps of sizes from 1 up to 10? From 1 up to 100? From 1 up to one million?

PRIME DESERTS ON THE NUMBER LINE

It is well known that there are strings of consecutive integers of any length you like on the number line with each number in the string a composite numbers.

For example,

$$6!+2 = 6\times5\times4\times3\times2\times1+2 = 722$$
$$6!+3 = 6\times5\times4\times3\times2\times1+3 = 723$$
$$6!+4 = 6\times5\times4\times3\times2\times1+4 = 724$$
$$6!+5 = 6\times5\times4\times3\times2\times1+5 = 725$$
$$6!+6 = 6\times5\times4\times3\times2\times1+6 = 726$$

is a string of five consecutive integers with the first number divisibly by 2, the second number divisible by 3, and so on, up to the fifth number divisible by 6. We have a string of five consecutive composite numbers. (Actually, $6!+1=721$ happens to be composite as well—but $6!+7=727$ is prime.)

We have found a "desert of primes" of length 5 on the number line and it will take a step of length more than five to pass over it.

Question: *The string 722, 723, 724, 725, 726 is not the first desert of length five. Which is?*

In general, the numbers

$$
\begin{aligned}
&(N+1)!+2\\
&(N+1)!+3\\
&\quad\vdots\\
&(N+1)!+(N+1)
\end{aligned}
$$

form a desert of primes of length N. Thus, there are prime deserts of every possible length within the number line. This means if you are walking to the right on the number line, stepping only one primes, but with steps bounded by some maximal size, you are sure to reach a desert you cannot cross.

PRIME DESERTS IN THE PLANE

In studying some questions in number theory, German mathematician Carl Friedrich Gauss (1777-1855) found it easier to think in terms of "whole complex numbers," that is, complex numbers of the form $a+ib$ with a and b integers. Opening up one's thinking to the complex realm often proves to be immensely helpful for attending to questions about real numbers. Today the set of "whole complex numbers" is called the set of *Gaussian Integers* in his honor.

We add and subtract and multiply Gaussian integers just as you would expect.

$$
\begin{aligned}
(a+ib)+(c+id)&=(a+c)+i(b+d)\\
(a+ib)-(c+id)&=(a-c)+i(b-d)\\
(a+ib)(c+id)&=(ac-bd)+i(bc+ad)
\end{aligned}
$$

Just as dividing two (real) whole numbers sometimes does and sometimes doesn't yield another whole number (for example, $20\div 5$ is a whole number, but $20\div 6$ isn't) dividing two Gaussian integers might or might not yield another Gaussian integer.

For example

$$\frac{14-5i}{2+3i}=1-4i$$

whereas

$$\frac{14-5i}{2-3i}=\frac{43}{13}-\frac{32}{13}i.$$

The first example shows that the Gaussian integer $14-5i$ is "composite:" it factors as $(1-4i)\times(2+3i)$.

The number $2+3i$ factors in essentially only four trivial ways.

$$
\begin{aligned}
2+3i&=1\times(2+3i)\\
&=-1\times(-2-3i)\\
&=i\times(3-2i)\\
&=-i\times(-3+2i)
\end{aligned}
$$

Challenge: *Prove that if*

$$2+3i=(a+ib)(c+id)$$

for integers a, b, c, d, then one of the factors is indeed 1, -1, i, or $-i$.

The numbers 1, -1, i, *or* $-i$ are considered "trivial" factors. We call a Gaussian integer *Gaussian composite* if it is the product of two non-trivial Gaussian integers and *Gaussian prime* otherwise.

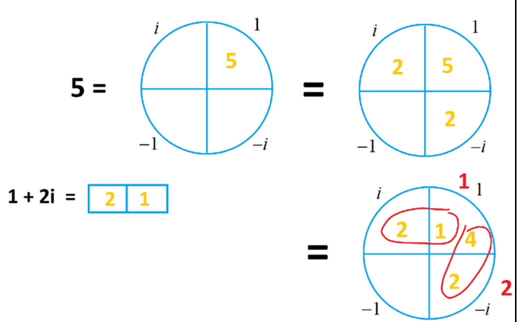
We have that $14-5i$ is Gaussian composite and $2+3i$ is Gaussian prime.

Challenge: *Show that* $1-4i$ *is also Gaussian prime.*

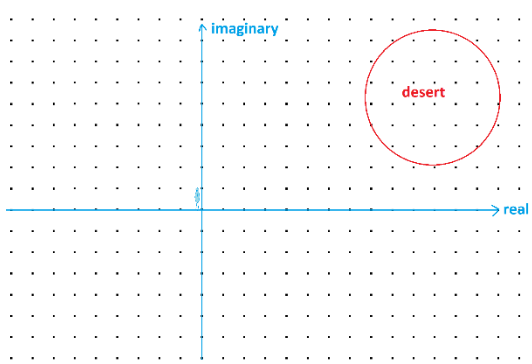
Curiously, a real integer can be prime among just the integers, but not prime when viewed as a Gaussian integer. For example, the prime 5 is Gaussian composite: $5=(1-2i)(1+2i)$.

Global Math Project ambassador Nick Johnson has developed an *Exploding Dots* way to perform Gaussian integer division.

Can you make sense of this computation of $\frac{5}{1+2i}$?



When mapped on the complex plane, the Gaussian integers correspond to the points with integer coordinates.



And mathematicians have proved, like the deserts of the primes on the number line, there are arbitrarily large regions of lattice points all of whose entries are Gaussian composites. That is, arbitrarily large prime deserts also exist in the complex plane.

However, the following is a famous unsolved problem in mathematics.

Starting at the origin, is it possible to leap from lattice point to lattice point in the complex plane, with all leaps smaller than some maximal size, and head off infinitely far from the origin, landing on Gaussian primes each and every time?

This problem was first posed by Basil Gordon in 1962.

Even though there are bigger and bigger prime deserts in the plane, it still might be possible to follow a path between and around them. No one knows if this is the case!

Also, it is known that there are infinitely many Gaussian primes along the real axis (and consequently, infinitely many on the imaginary axis too). But no one knows if there are any other lines in the plane that contain infinitely many Gaussian primes.

Optional Homework: *Solve either one of these unsolved problems. Your choice!*

CHALLENGE: Let $w=a+ib$ and $v=c+id$ be Gaussian integers. Set $r=\sqrt{c^2+d^2}$, the distance of v from the origin in the complex plane.

Draw a circle of radius $\frac{r}{\sqrt{2}}$ about w.

Prove that there is a multiple of v in the complex plane that sits on or inside this circle.

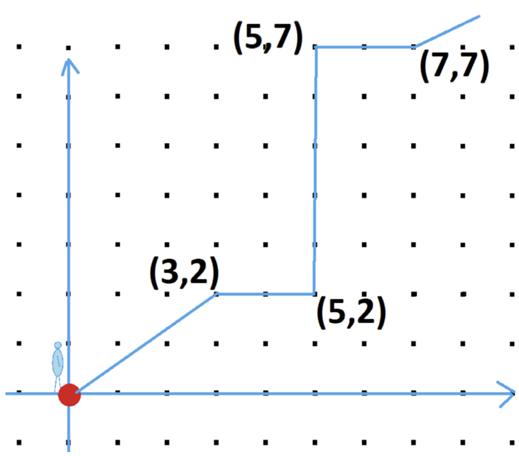
Hint: Imagine a picture of all the Gaussian integers in the complex plane as the points with integer coordinates. All the multiples of v form a tilted square lattice of these integer points, squares of side length r.

Prove that we can write $w = qv + d$ with q and d Gaussian integers with d within a distance $\frac{r}{\sqrt{2}}$ of the origin.

RESEARCH CORNER

Let's let go of complex number thinking and think of the lattice points in the ordinary Cartesian plane. These are the points (a,b) with each of a and b an integer.

i) Starting at the origin, is it possible to leap from lattice point to lattice point in the plane, with all leaps smaller than some maximal size, and head off infinitely far from the origin, landing each and every time on a point (a,b) with each of a and b a prime number?



ii) How about with $a+b$ a prime number?

iii) How about with at least one of a or b a prime number, but the direction one leaps in any one step must be different from the direction of the previous step?

iv) How about with a and b having only 1 as a common factor? And with changing direction from step to step? Or with no vertical or horizontal steps?

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