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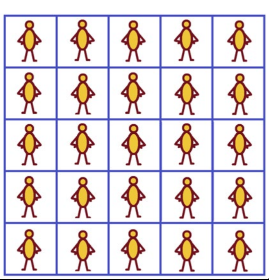
WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MARCH 2019

THIS MONTH'S PUZZLES

1. Some Students in a Grid
Twenty-five students stand in a large 5-by-5 grid drawn on the ground, one student per cell.



The students hope to shift positions to form a new arrangement of twenty-five students standing in the grid, one student per cell.

a) Can such a task be accomplished with each student taking a single left or right ("horizontal") or forward or back ("vertical") step to a neighboring cell?

b) Can such a task be accomplished with each student taking a single horizontal or vertical or diagonal step to a neighboring cell?

c) Can such a task be accomplished with each corner student staying fixed in place and each of the remaining 21 students taking a single vertical or horizontal step to a neighboring cell?

d) Can such a task be accomplished with some 8 students staying fixed in place and each of the remaining 17 students taking a single vertical or horizontal step to a neighboring cell?

2. More Students
One-hundred students stand in a 10-by-10 grid, one student per cell. They each take note of their left and right, forward and back neighbors.

The students then walk out of the grid and then back into the grid, one student per cell, making sure that each person has precisely the same set of neighbors as they had before (though the relative positions of their neighbors may have changed).

a) Is it possible that Albert moved to a cell two places to the left of his original position?

b) Prove that Betty not only has the same horizontal and vertical neighbors she had before, but she has the same diagonal neighbors too!

3. Even More Students
Three-hundred-and-sixty-one students stand in a 19-by-19 grid, one person per cell.

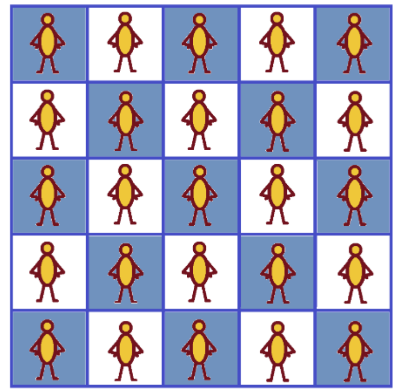
The students walk out of the grid and then back into the grid, one student per cell, making sure that each person has precisely the same set of horizontal and vertical neighbors as they had before.

Cathleen was originally in the center cell. Prove that she is in the center cell again.

GRID PARITY

The very first challenge is a classic puzzler.

The key to answering it is to color the grid in the pattern of a checkerboard.



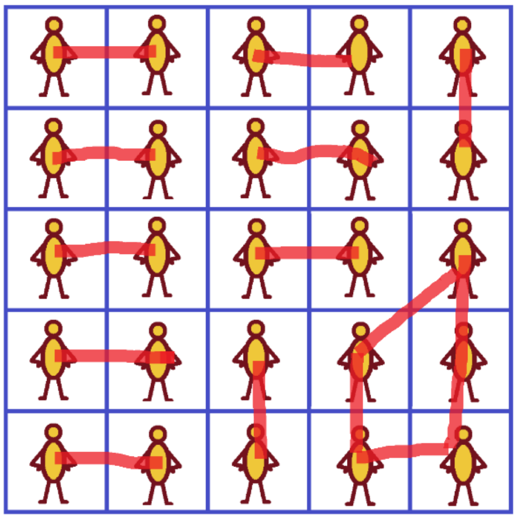
In the diagram shown we have 13 blue cells and 12 white cells. If we ask each student to take either a horizontal or vertical step to a neighboring cell, then each student in a blue cell is required to move to a white cell. (And vice versa, for that matter.) As there are more blue cells than white, it is impossible to accommodate all the students, one per cell, following the rules of puzzle 1a).

Comment: It is fun to actually try this exercise in practice with 25 students standing in a grid drawn on the ground with sidewalk chalk. Fun chaos results!

CHALLENGE: *Can the puzzle be solved if we select a student standing in a blue cell and allow her to stand still? Can the remaining 24 students then switch places by each taking a horizontal or vertical step to a neighboring cell, no matter which student we select to stand still in a blue cell?*

In general, if we have $N = ab$ students standing in the cells of an a-by-b grid, one per cell, it is impossible for them to switch places with single horizontal and vertical steps if a and b are each odd—the same coloring argument establishes this. If one value is even, then the puzzle can, for sure, be solved. (Why? Can you see how pairs of students can now just switch places amongst themselves?)

If the we allow students to take diagonal steps, then the first puzzle can be solved. In fact, it can be solved with just one student taking such a step. (Must it be a student in a blue cell?)



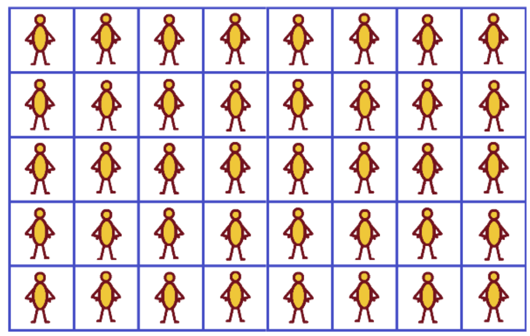
CHALLENGE: *Suppose we have $N = ab$ students standing in the cells of an a-by-b grid, one per cell, with a and b each odd. Can the puzzle always be solved with just one student taking a diagonal step?*

To attend to puzzles 1c) and 1d), suppose we allow an even number of students, say E of them, to stand still. Then we are really conducting a puzzle with $25-E$ students, an odd number of them, standing on cells in a checkerboard. The count of students in blue cells cannot equal the count of students in white cells, and so it is impossible for these students to switch color cells and still remain one student per cell. Puzzles 1c) and 1d) cannot be solved.

GENERAL RECTANGULAR REARRANGEMENTS

Suppose we have $N = ab$ students standing in an a-by-b grid, one student per cell. Suppose each student takes note of her horizontal and vertical neighbors, and then they all step out of the grid.

Is it possible for the students to reenter the grid to form a new arrangement of students, each with the same set of horizontal and vertical neighbors as before?



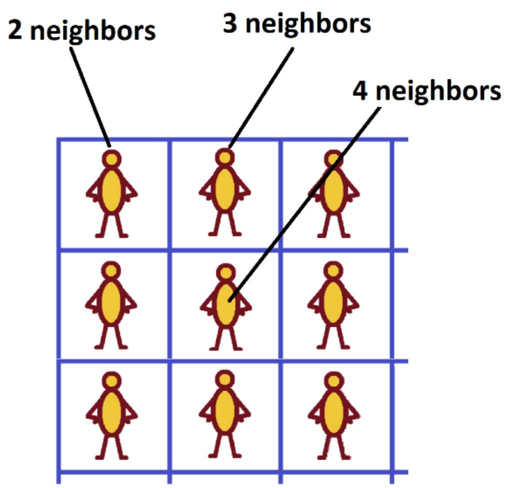
The answer is yes. For example, the students could reenter the grid to form a reflection of the arrangement they had before – reflect their original arrangement about a vertical or horizontal line of symmetry of the rectangular grid. Of, if they are in a square grid, they could copy a diagonal reflection of their original arrangement. They could also reenter to follow a 180° rotation of their original arrangement, or, if the grid is a square, a 90° rotation.

CLAIM: *These reflectional and rotational rearrangements are the only rearrangements that preserve the horizontal and vertical neighbor sets of each student.*

As no reflection or rotation of a 10-by-10 grid moves a student two places to the left (check this!), it is impossible for Albert to move two cells left. As each reflection and rotation preserves diagonal neighbors as well as vertical and horizontal ones, Betty is sure to have the same set of diagonal neighbors as she had before. In a 19-by-19 grid of cells, all reflections and rotations keep the center cell fixed in place and so Cathleen is indeed back in her center spot.

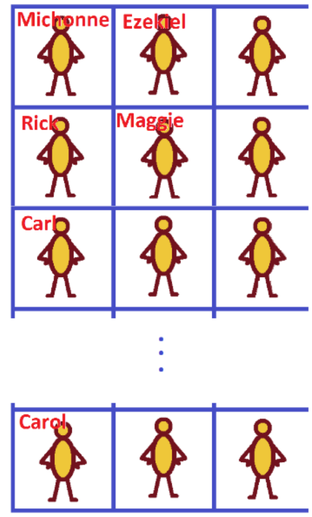
Let's prove the claim.

Each student in a corner cell has 2 horizontal or vertical neighbors; each "border" student has 3, and each "interior" student has 4.



So any rearrangement that preserves neighbor sets must have each corner student move to a cell with two neighbors, namely, another corner; each border student move to a cell with three neighbors, namely, another border cell; and each interior student to another interior cell.

So Michonne, in the top left corner, moves to another corner. By reflecting or rotating the grid, we can assume that she is actually back at the top left corner. But we have to note that whatever we conclude by following this assumption could be incorrect by some reflection or rotation of the grid.



Rick was originally on the second row, first column, in a border cell neighboring Michonne. He must move to a border cell still neighboring Michonne. There are two possibilities for this: back where he was or on the top row to Michonne's right.

Carl was originally on row three, column one, a border cell neighboring Rick. He must move to a border cell again neighboring Rick. Moving down this first column of people starting with Michonne in the corner, we reason that this same line of people moves either back to the same column with Michonne in the corner at top, or to the first row with Michonne in the corner to the left.

If the grid is square, then both options are possible: Carol, who was at the bottom left corner can either be at the bottom left corner again, or the top right corner. If the grid is rectangular, then only one option is possible: Carol must move to a corner and so must be back at the bottom left corner.

Either way, we can reflect across a diagonal line and assume this line of people do end up back in line along the first column. But we need to note that whatever we conclude from making this assumption could be incorrect by a diagonal reflection of the (square) grid.

But now Ezekiel, originally in the border cell to the right of Michonne must return to the only available border cell neighboring Michonne, namely, the same cell. And Maggie, originally in row 2, column 2, must be in the only available interior cell that neighbors Rick and Ezekiel, namely, the same cell. And so on.

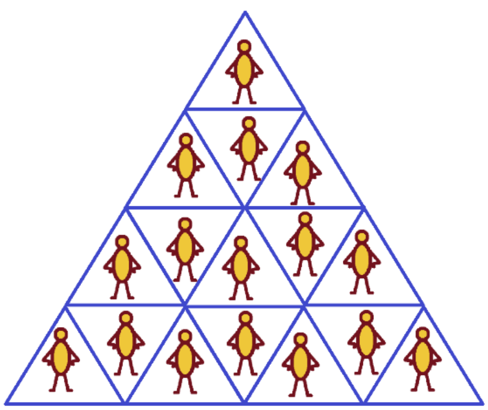
We argue now that all the folk in second column are fixed in place. And then we'll argue that all folk in the third column are fixed in place too. And so on.

We conclude … no one changed place!

But our conclusion is possibly incorrect: incorrect by a rotation or reflection of the grid. But this looseness is within the parameters of the claim!

RESEARCH CORNER

Examine rearrangements of students in triangular—and other—grids.



In graph theory: Imagine students standing at the vertices of different types graphs (trees, bipartite graphs, stars, wheels, complete graphs, etc.) and classify all rearrangments that preserve neighbor relationships.

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