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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

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TODAY'S PUZZLERS:

Consider the sequence

11212312341234512345612345671…

with its implied pattern continuing. This sequence is "doubly fractal" in that it contains copies of itself within itself in two fundamentally different ways.

(1) *Subtract one from each entry of the sequence and then delete any of the zeros that result. What remains is* 112123124123451234561…, *the original sequence!*

(2) *Each of the counting numbers* 1,2,3,… *appears infinitely often. If you delete the first occurrence of each counting number, what remains is* 112123123412345123456 1…, *the original sequence!*

a) Find another sequence satisfying these two properties.

b) Give an example of an infinite sequence of counting numbers satisfying property (1) but not (2). Find a sequence that satisfies (2) by not (1).

DOUBLY FRACTAL SEQUENCES

In the 1990s, mathematician Clark Kimberling studied sequences of counting numbers satisfying properties (1) and (2) and found a simple method for generating uncountably many of them.

Choose any non-zero real number θ and make an array of all the values $m+n\theta$ as m and n each run through the counting numbers 1, 2, 3, …..

For example, here is the top left portion of the array for $\theta = 0.5$.

m \ n	1	2	3	4	5	6
1	1.5	2	2.5	3	3.5	4
2	2.5	3	3.5	4	4.5	5
3	3.5	4	4.5	5	5.5	6
4	4.5	5	5.5	6	6.5	7
5	5.5	6	6.5	7	7.5	8
6	6.5	7	7.5	8	8.5	9

Whenever θ is rational, there will be repeat entries in the table (if $\theta = \frac{p}{q}$, then $1+(q+1)\theta = (p+1)+\theta$, for instance), and there will never be repeat entries if θ is irrational. (Prove this.)

Challenge: Prove that two distinct non-zero real numbers are sure to produce distinct arrays.

The entries in any row of the table are distinct and each is one more than its matching entry in the previous row. The entries in any column of the table are distinct and each entry is θ more than its matching entry in the column to the left.

To construct a doubly-fractal sequence, circle the smallest entry in the table, then the next smallest, then the next smallest, and so on. (If there are entries of repeat value, choose the leftmost one first.) In this way, we order the entries in the two-dimensional array.

The row numbers of that sequence of ordered entries form a doubly-fractal sequence!

To illustrate: For the array for $\theta = 0.5$, the red, yellow, green, blue, and then black circled entries represent the first five ordered terms of the array. They have, in turn, row numbers 1, 1, 2, 1, and 2.

	1	2	3	4	5	6
1	1.5	2	2.5	3	3.5	4
2	2.5	3	3.5	4	4.5	5
3	3.5	4	4.5	5	5.5	6
4	4.5	5	5.5	6	6.5	7
5	5.5	6	6.5	7	7.5	8
6	6.5	7	7.5	8	8.5	9

The row numbers of these first five ordered entries are, in turn, 1, 1, 2, 1, 2.

The doubly-fractal sequence that arises from $\theta = 0.5$ is

112121321321432144321….

Comment: If θ is irrational, there are no repeat entries in the array and the "when given a choice, choose the leftmost entry" protocol is not needed. One can avoid this protocol too for rational θ if you imagine this number as $\theta+\varepsilon$, where ε is a value satisfying $0<\varepsilon<2\varepsilon<3\varepsilon<\cdots<r$ for any positive real number r. (That is, ε is an *infinitesimal*: a quantity larger than zero but smaller than any real positive value.)

EXPLAINING THE PROCEDURE

It is not too tricky to see why Kimberling's approach produces doubly-fractal sequences.

First observe that if x and y are two entries in the table, with, say, y sitting r places to the right and s places down from x, then any two entries in the table with one sitting r places to the right and s places down of the other have the same order relation:

$x < y$ precisely if $x + r\theta + s < y + r\theta + s$.

	1	2	3	4	5	6
1	1.5	2	2.5	3	3.5	4
2	2.5	3	3.5	4	4.5	5
3	3.5	4	4.5	5	5.5	6
4	4.5	5	5.5	6	6.5	7
5	5.5	6	6.5	7	7.5	8
6	6.5	7	7.5	8	8.5	9

All entries in the same relative position have the same order relation.

It follows that if one were to ignore the first few rows and first few columns of the array to look at an (inifinte) sub-array, that sub-array has the same identical ordering pattern as the original array.

	1	2	3	4	5	6
1	1.5	2	2.5	3	3.5	4
2	2.5	3	3.5	4	4.5	5
3	3.5	4	4.5	5	5.5	6
4	4.5	5	5.5	6	6.5	7
5	5.5	6	6.5	7	7.5	8
6	6.5	7	7.5	8	8.5	9

In the shaded sub-array, the order of the entries is identical in structure to that ordering in the original array.
Here the ordered entries have row numbers 4,4,5,4,5,….

The sequence of row numbers of these terms, in order, is identical to original sequence of row numbers for the full array, except a constant value has been added to each term.

The full array for $\theta = 0.5$ gives the sequence

1121213213214321 4321….

But consider the sequence for this sub-array.

	1	2	3	4	5	6
1	1.5	2	2.5	3	3.5	4
2	2.5	3	3.5	4	4.5	5
3	3.5	4	4.5	5	5.5	6
4	4.5	5	5.5	6	6.5	7
5	5.5	6	6.5	7	7.5	8
6	6.5	7	7.5	8	8.5	9

This sub-array produces the same sequence with each entry increased by 1.

But we can also see this sequence as coming from tracing through the entries of the full array and not bothering to write down the row number 1 when it occurs. That is, ignoring all the 1s in the original sequence **1121213213214321 4321…** leaves the original sequence with each entry increased by 1. This leads to property (2).

Now consider the sequence for this sub-array.

	1	2	3	4	5	6
1	1.5	2	2.5	3	3.5	4
2	2.5	3	3.5	4	4.5	5
3	3.5	4	4.5	5	5.5	6
4	4.5	5	5.5	6	6.5	7
5	5.5	6	6.5	7	7.5	8
6	6.5	7	7.5	8	8.5	9

Since we are recording only row numbers, this sub-array produces the same sequence

1121213213214321432 1….

But we can also see this sequence as coming from tracing through the entries of the full array and not bothering to write down the first appearance of each row number when it occurs.

This gives property (1).

Challenge: The doubly-fractal sequences that arise from arrays this way for a given non-zero number θ satisfy a third curious property.

(3) *For any given number $a > 1$ in the sequence, look for the next occurrence of that number a in the sequence. Then each of the numbers 1, 2, 3, …, $a-1$ appears exactly once in the portion of the sequence between those two "consecutive" a s.*

For example,

11212132132143214321…
11212132132143214321…
11212132132143214321…
11212132132143214321…
112121321321432143215 43…

Prove this!

a) Let a, r be counting numbers, and k a counting number with $1 \le k \le a-1$. Prove that there is counting number $n > r$ so that

$$a + r\theta \le k + n\theta < a + (r+1)\theta .$$

(HINT: $a + r\theta = k + \left(r + \dfrac{a-k}{\theta} \right)\theta$.)

Use this to explain why the number k is sure to appear at least once between any two consecutive a s in the sequence.

b) Suppose we have

$$a + r\theta \le k + n\theta < k + m\theta \le a + (r+1)\theta$$

for counting numbers a, r, n, m with $1 \le k \le a-1$. Deduce that θ must be rational.

Use this to explain why the number k is sure to appear at most once between any two consecutive a s in the sequence.

Just recently, in their 2018 paper "A Novel Proof for Kimberlings' Conjecture on Doubly Fractal Seqences" (American Mathematical Monthly, 12:1), mathematicians Matin Amini and Majid Jahangiri proved that every sequence of counting numbers starting with the number 1 and satisfying properties (1) and (2) arises from an array for some non-zero real number θ. (Kimberling wondered if this was so.)

This completely characterizes doubly-fractal sequences.

We also see that any sequence starting with 1 satisfying properties (1) and (2) is also sure to satisfy property (3).

RESEARCH CORNER

1. What value θ generates the doubly-fractal sequence of the opening puzzle?

1121231234123451234561234567 1…

2. Consider the sequence

1 2 1 4 7 2 5 8 1 4 7 10 …

Here the nth term of this sequence is the result of writing n in base three, moving the first digit of that representation to the end, and determining which number has that base-three representation. For example, eleven in base three is 102. This is changed to 021, which is the base-three representation of the number seven. Thus the 11th number in this sequence is 7.

According to https://oeis.org/A048787 this sequence satisfies property (2) but not property (1) nor property (3).

Can you prove this?

3. The sequence

1 2 3 4 5 6 7 8 …

satisfies property (1), but not property (2) nor property (3).

Is there a sequence that satisfies property (3) but not property (1) nor property (2)?

Is there a sequence that satisfies properties (1) and (3), but not (2)? Is there a sequence that satisfies properties (2) and (3), but not (1)?

4. Let ω be the "first number" larger than any counting number. We have $1 < 2 < 3 < \cdots < \omega$.

What doubly-fractal sequence results if we choose $\theta = \omega$?

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