CURRICULUM INSPIRATIONS: www.maa.org/ci

Innovative Online Courses: www.gdaymath.com
Tanton Tidbits: www.jamestanton.com

WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JUNE 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

***

For teaching problem solving in the mathematics classroom check out the ever-growing MAA Curriculum Inspirations resources at www.maa.org/ci. Videos, essays, strategies, and, best of all, just plain, neat, cool, interesting math!

OPENING PUZZLE:
You are brought to a crime scene. You are told that a thief just made off with a bag full of diamonds, escaping on a bicycle. You come across the following pair of bicycle tracks in the snow, no doubt made by the fleeing thief. But which way did the thief flee?

Just by looking at the shapes of the tracks (tread marks, splashes of snow are inconclusive), can you determine which way the thieving cyclist went: left to right or right to left?

THE ANSWER:

Let's get straight into it. Let's answer the opening puzzler.

It makes sense that a bicycle leaves behind two tracks: one for the front wheel of the bicycle and the other for the back wheel. Can we say which track is most likely to be the back wheel track and which is the front wheel track?

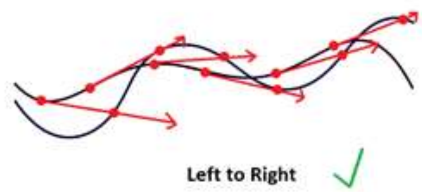
To answer this question, think about how a bicycle is built. The back wheel is fixed in its frame, unable to pivot, whereas the front wheel can turn freely. Thus we'll guess that the more stable track comes from the back wheel and the more wobbly track from the front wheel.

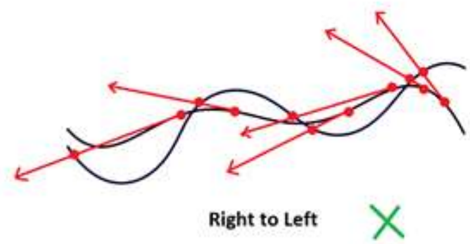
Now we are all set to determine which why the bicycle went. We just take this thinking one step further.

Look at a bicycle. Observe that the back wheel is not only fixed in its frame but also always points towards the point of contact of the front wheel to the ground - and at a fixed distance to boot (the distance between the axles of the wheels).

Loosely, then, we can say that each point on the back wheel track "should point towards the front wheel track at a fixed distance." More precisely, the tangent line at each point on the back wheel track should intercept the front wheel track at a fixed distance.

If you make a print of actual tracks on a large roll of poster paper (cover the wheels of a bicycle with sidewalk chalk, ride a wobbly path down the paper, and go over the traces, likely to be faint, with bright markers) you will see this in action. By holding up a yard stick to the tracks you indeed see that, in one direction, the tangent line to the back curve intercepts the front wheel track at some consistent distance – and that is not the case if you presume the motion is in the opposite direction.





We have, it seems:

For a given pair of bicycle tracks, it is not only possible to determine in which direction the bicycle traveled, but also the length of the bicycle that made those tracks.

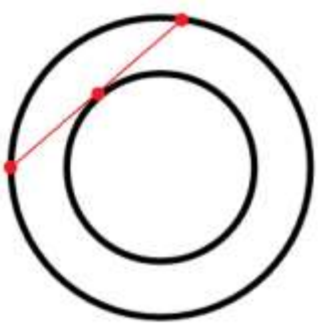
Making real bicycle tracks

RESEARCH CORNER

Actually, there is at least one instance in which we cannot determine the direction of motion of the cyclist:

If the rider rides a perfectly straight path, the two tracks of the wheels overlap along a single straight line. It is impossible in this case to ascertain the direction of motion from analyzing the tracks.

This is not the only example of rider motion for which we could not determine the direction of travel. Suppose a cyclists rides in a perfect circle. Because of the symmetry of the situation we cannot determine the direction of motion in this case too.



OPEN QUESTION: *Are these the only two cases in which we could not determine the direction of travel?*

More precisely:

If a pair of smooth curves have the property that each tangent line to one curve always intercepts the other curve at two locations, a fixed distance r either side of the point of contact along the tangent line, must the curves each have constant curvature?

I, personally, do not know the answer.

TWO OTHER OPEN QUESTIONS:

Could two bicycles of different lengths produce the same pair of (non-straight) bicycle tracks?

Could a bicycle produce a single non-straight track? (That is, is it possible to ride a non-straight path on a bicycle so that back wheel track covers the front wheel track? Alternatively, could a bicycle and a unicycle produce exactly the same non-straight track?)

Comment: All three questions can be united as one: *Could two bicycles of lengths r and s produce the same non-straight tracks?* (Allow the cases $s=-r$ and $s=0$ too.)

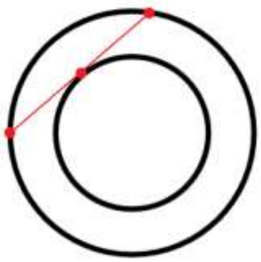
These open questions are fun to explore, but they may require some technology help in order to draw bicycle tracks with the correct mathematical structure. (Write a set of parametric equations for a back wheel curve, compute the unit tangent vector at a location on this curve and plot the endpoint of this vector. The trace of this point is the front wheel track. I constructed the curves in the opening puzzler this way.)

Reference: I first learned of this problem in Joseph Konhauser, Dan Velleman, and Stan Wagon's text *Which Way did the Bicycle Go? … and Other Intriguing Mathematical Mysteries* (MAA, 1996). They learned of the problem from a geometry course being developed at Princeton in the 1980s.

This problem is described in Sherlock Holmes novel too, but Doyle, according to the three authors above, gives an incorrect (non-mathematical) solution for determining the direction of travel.

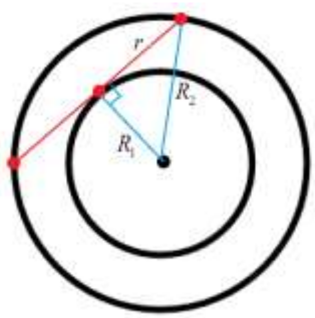
GOING FURTHER:
THE AREA BETWEEN TRACKS

Back to the pair of circular tracks.



Question: *If the distance between the two bicycle wheels that made this pair of tracks is r units, what is the area between the two tracks?*

Answer: Surprisingly the answer is πr^2 no matter the sizes of the concentric circles.



Let R_1 and R_2 be, respectively, the inner and outer radii of the concentric circles.

A tangent line to a circle is perpendicular to the radius of the circle at the point of contact. In our diagram we thus see a right triangle and the Pythagorean Theorem gives $R_1^2 + r^2 = R_2^2$.

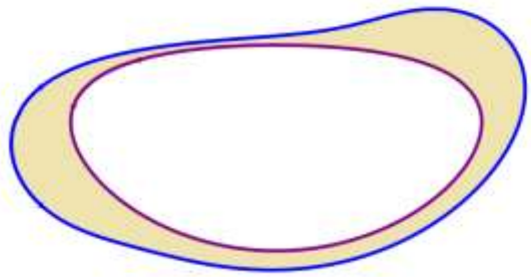
The area between the two circles, that is, between the two tracks, is:

$$\begin{aligned} \text{area} &= \pi R_2^2 - \pi R_1^2 \\ &= \pi\left(R_2^2 - R_1^2\right) \\ &= \pi r^2. \end{aligned}$$

A BOLD CLAIM:

Suppose we ride a bicycle in a large convex loop. I claim:

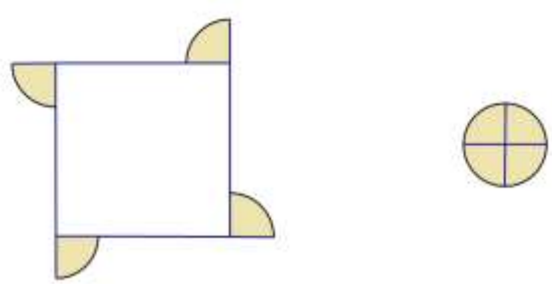
The area between these tracks is again sure to be πr^2. The area between a pair of bicycle tracks is a universal invariant.



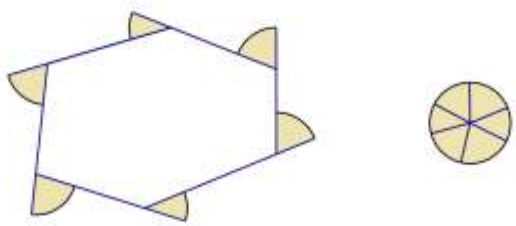
This is hard to believe! Each night on my evening neighborhood ride, no matter which loop through the streets I choose to follow, the area between my two tracks is universally fixed? Freaky!

(By the way, the above pair of curves pictured were generated with computer software and are mathematically correct tracks. Can you see which way around the loop the bicycle traveled?)

Understanding the claim:
Consider riding the bicycle in a polygonal loop. For example, force the back wheel of the bicycle to follow a square path, pivoting the front wheel about its corners. Can you see that the area between the back and front wheel tracks is certainly πr^2 ?

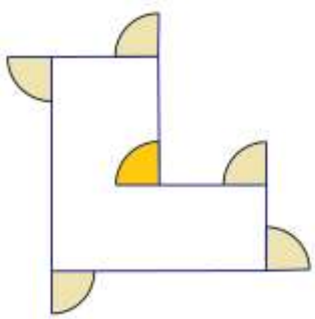


In general, if we force the back wheel to follow a convex polygonal path, the region between the front wheel and back wheel tracks is composed of a set of sectors. Because the exterior angles of a regular polygon sum to 360°, these sectors fit together perfectly to make one full circle.



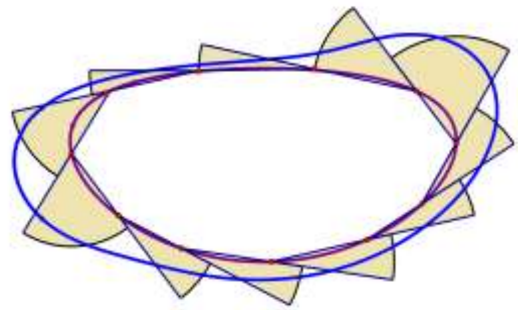
The area between the front-wheel and back-wheel tracks for a cyclist riding in any convex polygonal path is certainly πr^2.

Side Exploration: Is the same true for a cyclist riding a non-convex polygonal paths?



The answer is yes if we regard areas swept out in a counter-clock and clockwise directions as having opposite signs.

Now, any smooth convex curve can be approximated as a convex polygon by drawing short line segments between points of the curve. And if we ride the bicycle along the convex polygonal approximation we know the area between the bicycle tracks approximating the original tracks is sure to be πr^2.



We can get a better approximation of the true area between the tracks by using a finer polygon (many more sides of shorter lengths) to approximate the back wheel track. But we know the area between the tracks in this approximation will be πr^2 too.

Since all approximations to the true area between the tracks have value πr^2 and these approximate values converge to the area we seek, we can only conclude that the area between the original tracks is sure to be πr^2.

Comments: If one rides a non-convex loop, then the area between the tracks is again πr^2, as long as one counts area swept out in opposite directions (clockwise or counter-clockwise) as opposite in sign.

Traversing a convex loop has you undergo one full turn overall, and the area between the tracks is πr^2. If you ride a double loop, or a triple loop, then the area between the tracks you create will be $n\pi r^2$, where n is the number of full turns you complete overall. The overall effect of riding a figure eight is no turning and the area between the tracks in this case (counted with sign) is zero.

Question: Knowing this fact about the area between bicycle tracks, have you any partial thoughts towards answering some of the previous open questions?

tanton.math@gmail.com