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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JULY 2020

The Global Math Project team is putting together a book: *Puzzles Explained by Exploding Dots.* The topic of this essay is a one puzzle recently shared with me that I realized fits well in this category. It's called *The Devil's Chessboard* and it has apparently been circulating through the mathematics community at least a decade.

Enjoy!

THIS MONTHS' PUZZLER:

You and a colleague know you are soon to play the following "game." You each understand the rules of the game and you have some time to converse together about your strategy for possibly winning the game.

This evening, Mrs. X. will come fetch you, just you, and lead to a room. You will not see or be able to communicate with your colleague in any way from this point forward.

In the room is a single table on which sits an 8-by-8 chessboard and a pile of 64 identical coins.

Mrs. X. will place the coins on the board, one per cell, choosing at random as she goes along whether a particular coin sits heads up or tails up. You will be permitted to watch her do this.

T	H	H	T	T	T	H	T
T	H	H	H	T	H	T	H
T	H	T	T	H	T	H	T
H	T	T	H	H	T	T	T
T	H	T	H	T	H	H	H
H	T	H	H	H	H	T	T
T	H	T	T	H	H	T	T
T	H	H	T	H	T	H	H

When done, she will then point to a coin somewhere on the board and says: "This is the favored coin." Of course, there is nothing to distinguish one coin from another, but Mrs. X. will have you take note of one particular coin on the board.

Then Mrs. X. will have you flip *one* coin in the board: to change it from heads up to tails up, or vice versa. When done, you will then be taken away to wait alone in a separate room.

While you wait, your colleague will be brought into the room with the board and simply told to pick up the favored coin. She may touch and pick up one coin only and it must be the correct favored coin.

What strategy can you both devise so that your colleague will know the correct coin to choose?

INITIAL THOUGHTS

Coins can be in one of two states, heads up or tails up, and this, along with the fact that the count of coins is power of two ($64 = 2^6$) is a little suspicious. Might this puzzle be making use of binary representations of numbers? Can we use of a $1 \leftarrow 2$ Exploding Dots machine?

It seems natural to number the cells of the chess board 1 through 64, say, starting at the top left cell of the board, reading left to right, top to bottom, so that that the bottom right cell is number 64.

Is it helpful to write the cell numbers in binary: 1, 10, 11, 100, 110, …., 1000000?

Should we focus on the location of the heads-up coins on the board? A flip of single coin will either add or remove a coin from this set. How might we be able to use this to communicate a specific cell number?

GIVING THE PUZZLE AWAY

It is actually better to number the cells 0 to 63 from the top left cell to the bottom right cell, and to represent these numbers as six-digit binary numbers. (Include leading zeros.)

000000, 000001, 000010, ….,
…, 111110, 111111.

This feels "balanced" in that every six-digit sequence of 0s and 1s appears in this list. (Working with the numbers 1 through 64 is "unbalanced.")

Call a six-digit binary number a "heads number" if the coin in its cell position is heads. There could be anywhere from zero to sixty-four heads numbers.

From the set of heads numbers, create another six-digit binary number as follows:

1. Write out the heads numbers, one under the other.
2. Under each column, write a "1" if the count of 1s in that column is odd and write a "0" if it is even. (If there are no heads numbers, that is, if Mrs. X. lays out nothing but tails up coins, write 000000.)

For example, suppose Mrs. X. lays out heads on cells 2, 11, 17, 22, 46, and 62, and all other cells are tails. Then we create from this set the code 011110.

2 =	0	0	0	0	1	0
11 =	0	0	1	0	1	1
17 =	0	1	0	0	0	1
22 =	0	1	0	1	1	0
46 =	1	0	1	1	1	0
62 =	1	1	1	1	1	0
	even	odd	odd	odd	odd	even
	0	1	1	1	1	0

Now we have the opportunity to change this code by adding another heads coin to the set or to take one out. Can we change the code to match the cell number of the favored coin?

Yes!

Following the example, suppose the favored coin is in cell 53, which has code 110101. Then we need to change the parity (evenness/oddness) of four columns.

Change 0 1 1 1 1 0

to 1 1 0 1 0 1

Consider the coin in the cell with the number that has a 1 in each of the columns we need to change and 0 in each of the columns we don't need to change. In our example, that's cell number 101011, which is 43.

If we add this coin to the list of heads coins (by turning coin 43 from tails to heads) or remove this coin from the list (by turning coin 43 from heads to tails), we change the count of 1s in each column we need to change by one, and don't change the count of 1s in the columns that we don't need to change. Either way, by flipping coin 43, we get a new set of heads coins that communicates the number of the coin in the favored cell!

2 =	0	0	0	0	1	0	
11 =	0	0	1	0	1	1	
17 =	0	1	0	0	0	1	
22 =	0	1	0	1	1	0	
46 =	1	0	1	1	1	0	
62 =	1	1	1	1	1	0	
	even	odd	odd	odd	odd	even	
43 =	1	0	1	0	1	1	
	odd	odd	even	odd	even	odd	
	1	1	0	1	0	1	**= 53**

In this way we can communicate to our colleague any desired cell number from any collection of heads coins Mrs. X lays out!

To summarize the procedure:

1. Look at the positions of the coins showing heads.

2. Write the cell number of each of these coins as a six-digit binary code from 000000 to 111111. Place these numbers in a vertical list.

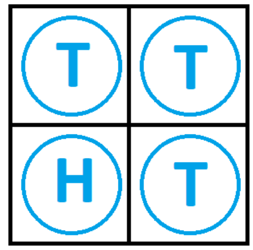
3. For each column write 0 if the count 1s in that column is even, write 1 otherwise. This gives an "auxiliary" six-digit code.

4. Compare the auxiliary code with the six-digit code of the cell number of the favored coin. For each position along that code, write 1 if there is a mismatch of digits, write 0 otherwise. This gives yet another six-digit code. This is the code of the cell number of the coin you flip. You've now arranged matters so that the heads on the board encode the cell number of the favored coin.

5. When your colleague enters the room, she conducts steps one, two, and three. She picks up the coin with cell number the auxiliary code she computes.

TRY IT!

Try this game with the smaller example of a two-by-two board. Here the cell numbers are the two-digit binary codes 00, 01, 10, and 11.



Does practicing with this small example help make sense of the procedure?

RESEARCH CORNER

Every solved problem, of course, is an invitation to explore and play more.

Is there a base six version of this puzzle?

Suppose Mrs. X. laid a die on each cell of a board (what dimensions?), each showing one of the numbers 1, 2, 3, 4, 5, and 6. (Perhaps regard "6" as the same as zero?)

Could you change the showing face of one—or maybe five?—dice to communicate a particular cell of the board?

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