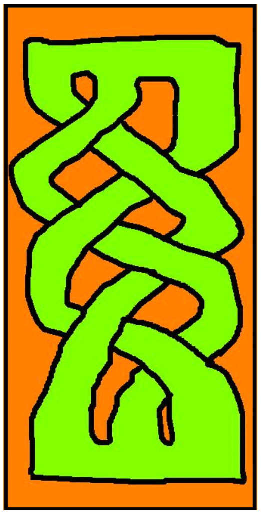


CURRICULUM INSPIRATIONS www.maa.org/ci

Innovative Online Courses www.gdaymath.com
Tanton Tidbits www.jamestanton.com

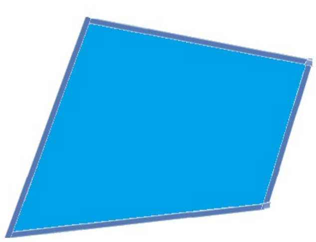
Global Math Project www.globalmathproject.org

WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

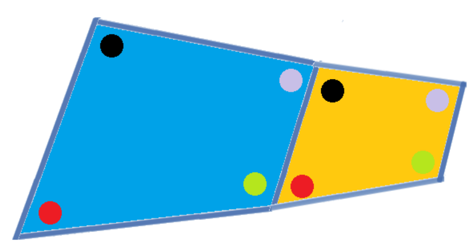
MAY 2019

TODAY'S PUZZLER
Draw a quadrilateral with no two opposite edges equal in length.

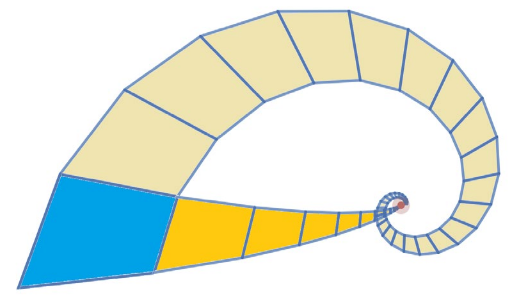


Choose a pair of opposite sides and on the shorter of the two, attach a (reduced) scaled copy of the quadrilateral.

(The dots in my picture show matching angles.)



Keep doing this over and over again. And do it over and over again for the other pair of opposite edges too.



Prove that the two "arms" one creates are sure to converge to a single point in the plane!

THE ORIGIN OF THE PUZZLER

Daniel Scher shares this geometry gem in his March 2019 *Sine of the Times* blog post. He explains that it is originally due to his former Cornell University undergraduate thesis advisor, David Henderson, sadly recently deceased. Daniel challenges readers to prove the result.

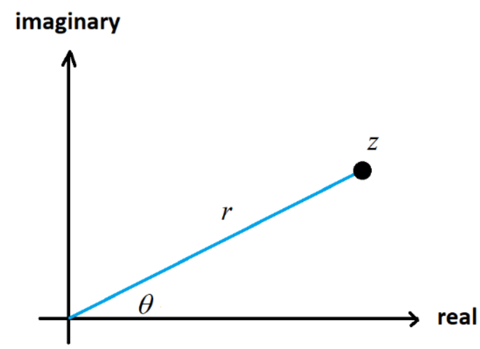
And in the comments section of the post two readers do. Further, "Eric," shares a dynamic DESMOS demonstration for all to play with. (I used this demonstration to create the puzzler image.)

I am charmed by this result and would like to present a proof of it. The proof here follows the same lines of the one given by "Josh H" in the post.

RECALLING SOME COMPLEX NUMBER GEOMETRY

In the May 2019 essay I gave a (mighty swift) overview of the use of complex numbers in geometry.

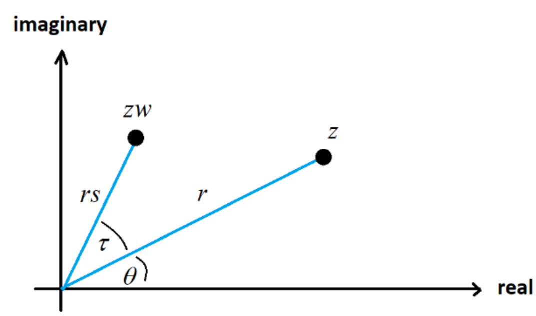
We saw there that each complex number z can be written in the form $z = re^{i\theta}$ where r (its *magnitude*) is its distance from the origin in the complex plane and θ (its *argument*) is its angle of elevation from the positive real axis.



Multiplying z by a complex number $w = se^{i\tau}$ changes its angle of elevation to $\theta + \tau$ and changes its distance from the origin to rs:

$$zw = re^{i\theta} \cdot se^{i\tau} = rse^{i(\theta+\tau)}.$$

That is, multiplication by w has the geometric effect of rotating each point in the complex plane counterclockwise about the origin though angle τ and pushing it to or from the origin by a factor s.



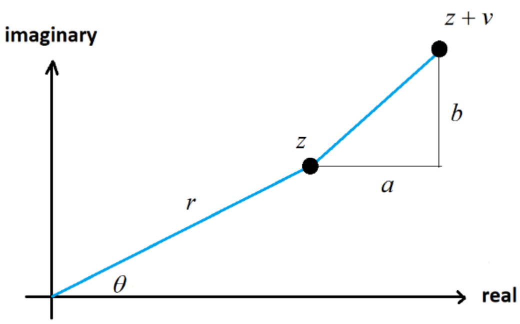
If $w = se^{i\tau}$ with $0 < s < 1$, then the sequence of complex numbers

$$z, \; zw, \; zw^2, \; zw^3, \; zw^4, \; \ldots.$$

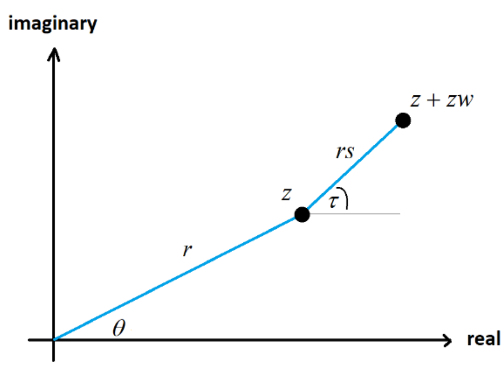
decrease in magnitude and "spiral" around the origin to, in fact, converge to the origin.

On another note …

If a complex number v is a units to the right of and b vertical up from the origin, then the complex number $z + v$ is a units to the right of and b vertical up from z.



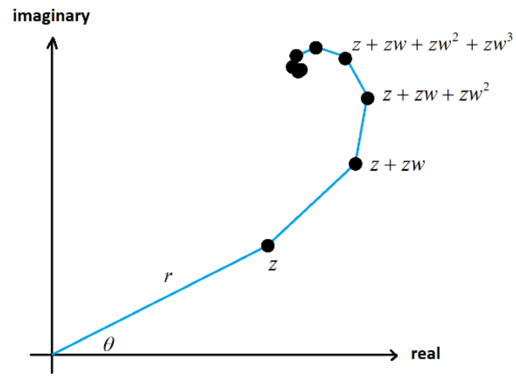
In particular, with z and w as before, $z+zw$ is the point in the complex obtained by "adding" to the line segment from 0 to z that line segment contracted by a factor s and rotated through an angle τ .



The point $(z+zw)+zw^2$ "adds" to this a line segment contracted by s twice and turned through a second angle τ . And so on. Thus, the sequence of complex numbers

$$z$$
$$z+zw$$
$$z+zw+zw^2$$
$$z+zw+zw^2+zw^3$$
$$\vdots$$

gives an "arm" of segments, each segment scaled down in length by a factor of s compared the previous segment and rotated through an angle τ .

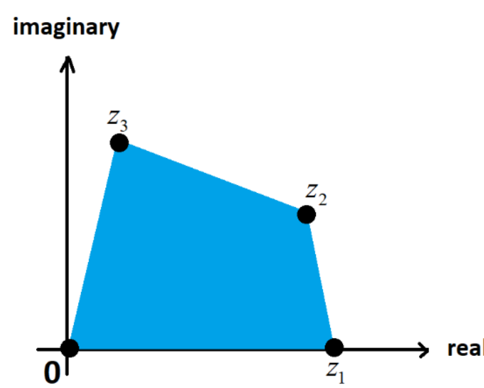


By the geometric series formula, this sequence of complex numbers converges to the complex number

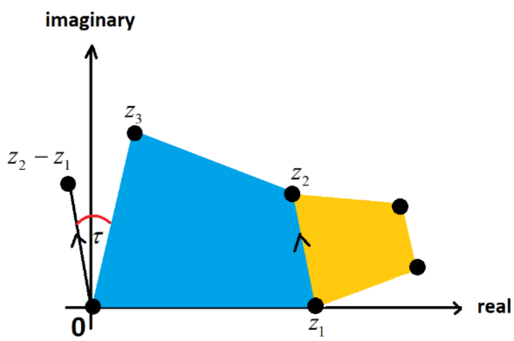
$$z\left(1+w+w^2+w^3+\cdots\right)=z\cdot\frac{1}{1-w}$$
$$=\frac{z}{1-w}.$$

PROVING HENDERSON'S RESULT

Let's assume the original quadrilateral is situated in the first quadrant of the complex plane with one corner at the origin and remaining corners at positions z_1 , z_2 , and z_3 as shown. Let's also assume that the distance between z_1 and z_2 is less than that between 0 and z_3 , and that the distance between z_3 and z_2 is less than that between 0 and z_1 .



Consider a scaled copy of the quadrilateral (in yellow) attached to its right edge from z_1 to z_2.



The line segment from $z_1 - z_1 = 0$ to $z_2 - z_1$ is a translated copy of this right edge. We see that the new quadrilateral can be constructed by 1) rotating the original quadrilateral about the origin and scaling it so that the edge from 0 to z_3 aligns with the segment from 0 to $z_2 - z_1$, and then 2) translating this rotated and scaled copy so that the point 0 moves to z_1.

Let τ be the measure of the angle shown and let s be the ratio of the length from 0 to $z_2 - z_1$ to the length from 0 to z_3. Here $0 < s < 1$. Set $w = se^{i\tau}$.

Then multiplication by w rotates all points in the plane about the origin by angle τ and moves them close to the origin by a factor s. The addition of z_1 then translates all points in the plane. These two actions match actions 1) and 2) just described.

Thus the transformation

$$u \in \mathbb{C} \;\mapsto\; uw + z_1 \in \mathbb{C}$$

is the transformation that takes the points of the original quadrilateral to the points of the desired scaled copy on its right edge.

But what complex number is w?

Our transformation takes the point z_3 in the original quadrilateral to the point z_2 in the desired scaled copy. So we must have

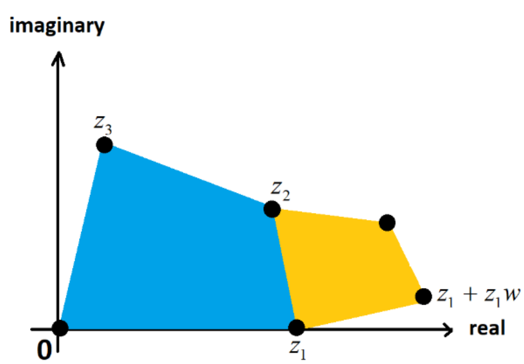
$$wz_3 + z_1 = z_2.$$

This shows

$$w = \frac{z_2 - z_1}{z_3}.$$

(One could have deduced this directly: Multiplication by any complex number w corresponds to rotation and scaling about the origin. And clearly multiplication by $\frac{z_2 - z_1}{z_3}$ takes z_3 to $z_2 - z_1$, so this must be the value of w we need.)

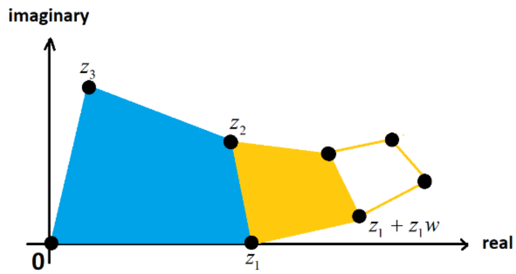
Notice that the point z_1 in the original quadrilateral is taken to $z_1 w + z_1$ by this transformation.



Continuing the right arm

Let's continue to label the points on the bottom edge of each quadrilateral we construct. So far we have the points z_1 and $z_1 + z_1 w$ on the bottom edge of the first quadrilateral in the arm.

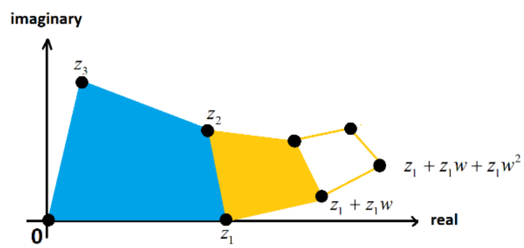
The second quadrilateral in this right arm is a copy of the original (blue) quadrilateral rotated by an angle 2τ, scaled by s^2, and translated by $z_1 + z_1 w$.



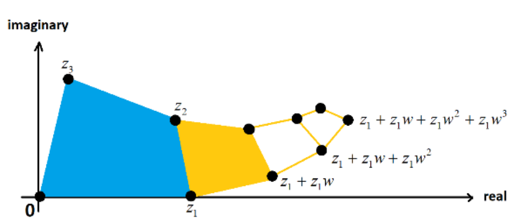
It is the image quadrilateral under the transformation

$$u \in \mathbb{C} \ \mapsto \ uw^2 + \left(z_1 w + z_1\right) \in \mathbb{C}.$$

In particular, the label of the next point along the bottom edge is
$z_1 w^2 + \left(z_1 w + z_1\right)$.



The third quadrilateral of the arm is a copy of the original (blue) quadrilateral rotated by 3τ, scaled by s^3, and translated by $z_1 + z_1 w + z_1 w^2$. The label of the next point along the bottom edge is
$z_1 w^3 + \left(z_1 + z_1 w + z_1 w^2\right)$.



And so on.

We know from our previous work that these bottom edge points converge to $\dfrac{z_1}{1-w}$. As the areas of the quadrilaterals along the arm converge to zero, all matching points from quadrilateral to quadrilateral in the arm also converge to the same point $\dfrac{z_1}{1-w}$ in the plane.

And we know what w is. It's $\dfrac{z_2 - z_1}{z_3}$.

We have

The arm of quadrilaterals constructed on the right side of the quadrilateral converge to the point

$$\frac{z_1}{1-w} = \frac{z_1}{1 - \dfrac{z_2 - z_1}{z_3}} = \frac{z_1 z_3}{z_1 + z_3 - z_2}.$$

Now let's conduct the same analysis for the scaled rotated copies of the quadrilateral that produce the arm on the top edge of the original quadrilateral. The analysis will be identical, with roles of the two axes essentially interchanged. (We did not need z_1 to lie on the axis in our work above.) In particular we see that z_3 will be playing the role of the point z_1, and z_1 will be playing the role of z_3, and 0 and z_2 continue to "play themselves." So without any work we deduce

The arm of quadrilaterals constructed on the top side of the quadrilateral converge to the point

$$\frac{z_3 z_1}{z_3 + z_1 - z_2}.$$

And this is the same point! We have proved the claim of the puzzle.

RESEARCH CORNER

1. What variations can result if one or both pairs of opposite edges of the initial quadrilateral have equal length?

2. What if each quadrilateral is rotated and scaled and then attached to the shorter edge with flipped orientation? Does one obtain arms that still converge to a common point?

3. Is there a three-dimensional version of this result?

First, is there a three-dimensional solid with six polygonal faces with each opposite pair of faces similar in shape but of different areas? If so, what happens when one constructs "arms" of solids on each of the faces of smaller area? Do the three arms converge to a common point in space?

tanton.math@gmail.com