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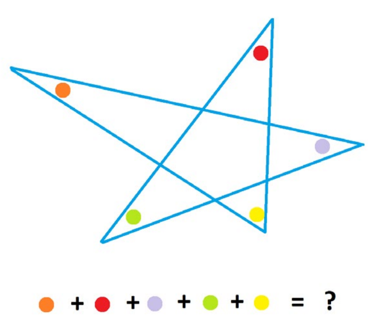
WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JANUARY 2021

THIS MONTHS' PUZZLER:
It's a classic! Draw a wonky five-pointed star. What is the sum of the angle-measures of its five tips?

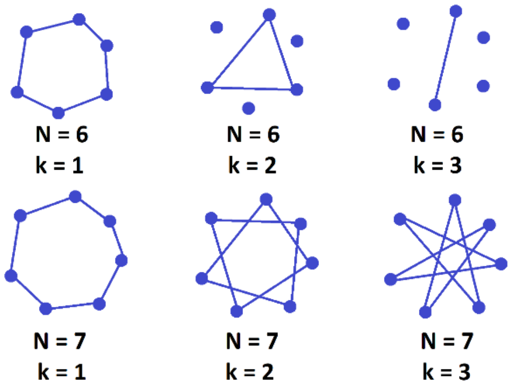


MAKING STARS

Draw N dots on a circle. They need not be evenly spaced. Starting at one dot, draw a line segment to the dot k places clockwise from it. And from that dot, draw a line segment to the dot k places clockwise from it. And so on. One will eventually return the starting dot and thus create a picture of a star. (Is it obvious that one first returns to the starting dot?)

Let's call the picture that results an (N,k) star.

We'll assume that $k \le \frac{N}{2}$. If k is larger, then just consider the picture as being constructed in a counter-clockwise direction by connecting dots a smaller distance apart.



If N and k have a common factor, say, their greatest common factor is $d > 1$, then only every d th dot is reached with line segments. Let's assume N and k are such that $d = \gcd(N,k) = 1$. Then each and every dot in the ring is the tip of the star. Let's call such a diagram "complete."

> If $\gcd(N,k) = d$, then $\frac{N}{d}$ and $\frac{k}{d}$ are both integers and they share no common factor. The (N,k) star is actually a complete $\left(\frac{N}{d},\frac{k}{d}\right)$ star. For instance, a $(6,2)$ star is, in reality, a $(3,1)$ star.

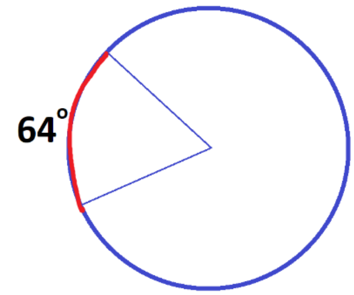
An $(N,1)$ star is a polygon with N sides.

Akin to what the opening puzzler asks, let's see if we can determine a general formula for the sum of angle measures in a complete (N,k) star.

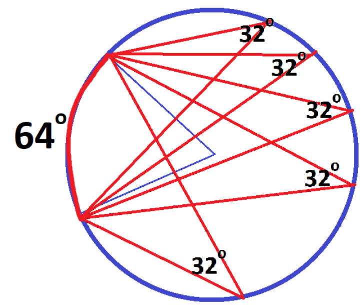
We do know, at least, from the geometry class that the angles in a $(N,1)$ star have measures summing to $(N-2)180^\circ$.

SOME GEOMETRY

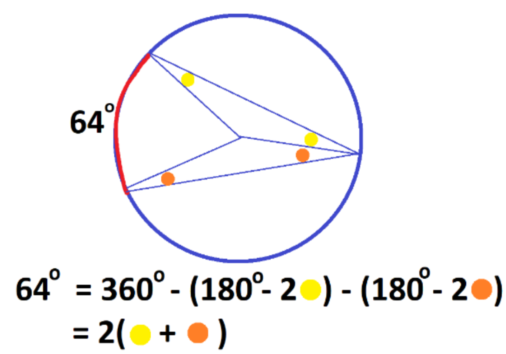
Each arc of a circle represents a measure of turning.



It is a surprise to learn that any angle drawn from a given arc to a point on the circle has measure exactly half this amount of turning.



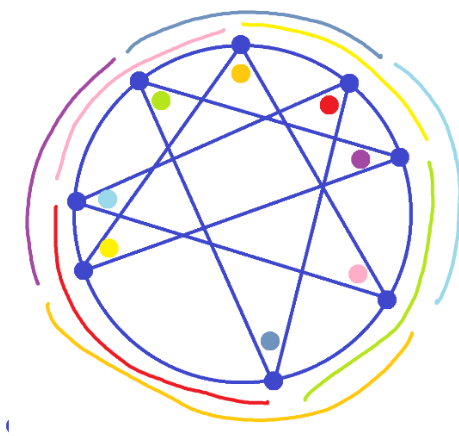
One proves this by drawing in extra radii and chasing common angles in isosceles triangles. The next picture illustrates one scenario to consider. All other scenarios are established similarly.



ANGLES IN STARS

We can now see a path forward. Look at the complete $(8,3)$ star as an example. We see that each angle at a tip has measure half the arc of the circle, and that each arc is "covered" by two tip angles. Thus, the sum of angle-measures of an $(8,3)$ star must be

$$\frac{1}{2}\times\left(2\times 360^\circ\right)=360^\circ.$$



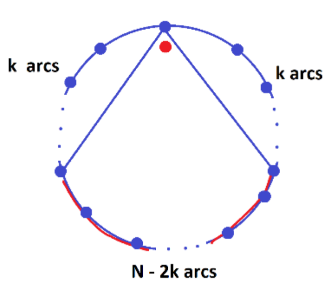
N = 8
k = 3

In general, for a complete (N,k) star, the angle-measures at the tips must sum to

$$\frac{1}{2}\times\left(r\times 360^\circ\right)=r\times 180^\circ$$

where r is the count of times an arc is covered by some angle at a tip. (By the symmetry of matters, each and every arc has the same value for r.)

In a complete (N,k) star, each angle at a tip covers $N-2k$ arcs between dots.



Consequently, each arc between two dots is covered by $N-2k$ tip angles.

Thus the sum of angles in a complete (N,k) star is $(N-2k)180^\circ$.

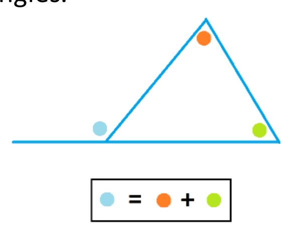
If an (N,k) star is not complete, then it is equivalent to a $\left(\frac{N}{d},\frac{k}{d}\right)$ star where $d=\gcd(N,k)$, and so has sum of angle measures

$$\frac{N-2k}{d}\cdot 180^\circ.$$

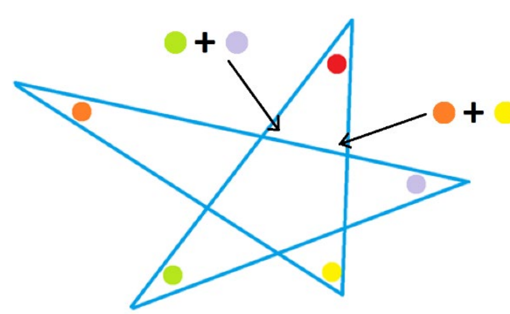
THE OPENING PUZZLER

The five-star puzzler does not assume that the points of the star lie on a circle. If this were a proper $(5,2)$ star, the angles at its tips would have measures summing to 180°. Does a lopsided star have this same angle sum?

A standard geometry result states that "the exterior angle of a triangle has measure the sum of the measures of its two remote interior angles."

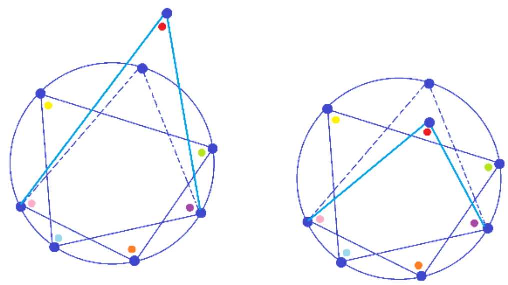


(One establishes by noting the measure of the unlabeled angle can be computed two different ways.) This allows us to see that the angles in a five-pointed star do indeed always have measures summing to 180°.



RESEARCH CORNER

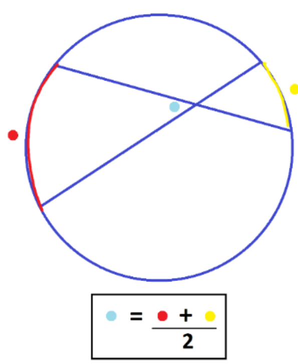
Show that shifting one point of a (N,k) off from the circle slightly does not change the sum of angle measures of the tips of the star.



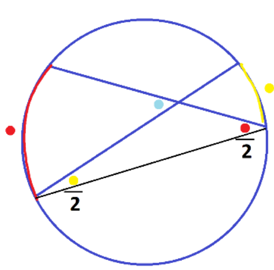
How slight does "slight" need to be? Can you now develop a general argument giving the sum of angle measures of any (N,k) star whose tips do not necessarily lie on a circle?

Another Result from Geometry

We have this result.

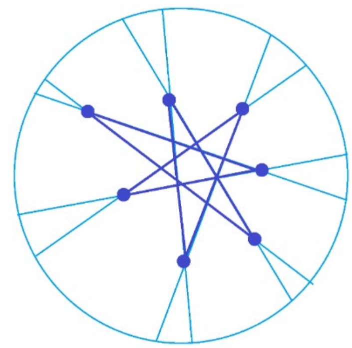


One proves this by drawing in one chord and using the two geometry results stated in this essay.



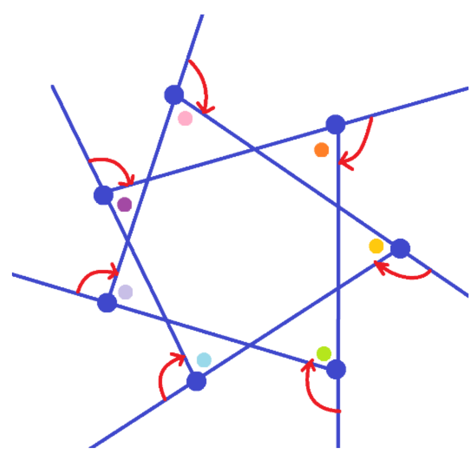
Is it always possible to enclose a wonky (N,k) diagram within a circle and use this third geometry result to analyze the sum of angle measures?

For instance, in this picture I can see that the sum of angle measures of the tips of the dark blue star must be $\frac{1}{2}\cdot 360^\circ = 180^\circ$.



Another Possible Approach

Can you prove that a complete (N,k) star has "winding number" k? Can you prove that the exterior angles of the star thus sum to $k \times 360^\circ$? Can you then use this to establish that the sum of angle measures we seek is $180N - 360k$ degrees?



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