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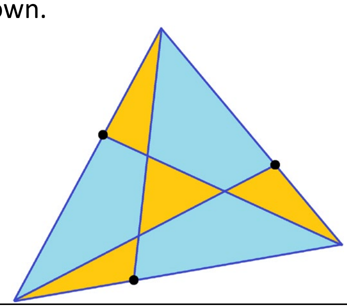
WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JANUARY 2019

A TRIANGLE PROBLEM:
Three lines in a triangle, each emanating from one vertex and intercepting its opposite side, create four interior triangles and three interior quadrilaterals as shown.



If the four triangles have the same area and the three quadrilaterals have the same area, what interesting things can you say about the ratios at which these lines intercept the sides of the triangle?

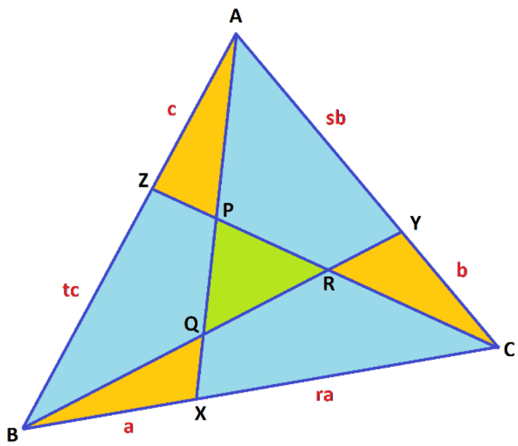
RECALLING ROUTH'S THEOREM

In the February 2016 essay, we discussed and proved Edward Routh's 1896 result about the area of an inner-most triangle formed by three Cevians.

Routh's Theorem: *If three Cevians of a triangle intercept the sides in* $r:1$, $s:1$, *and $t:1$ ratios, then the innermost triangle formed by them has area the fraction*

$$\frac{Area\ \Delta PQR}{Area\ \Delta ABC} = \frac{(rst-1)^2}{(1+r+rt)(1+s+sr)(1+t+ts)}$$

of the area of the whole triangle.



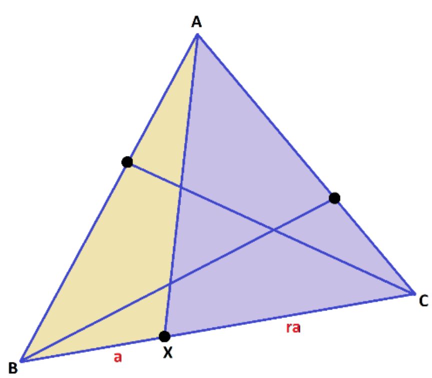
Let's call the area of the whole triangle 1 unit of area.

Since ΔBXA has the same height as the whole triangle but base the fraction $\frac{a}{a+ra}$ of its base, we have

$$Area\ \ \Delta BXA = \frac{a}{a+ra}\cdot 1 = \frac{1}{1+r}.$$

Similarly,

$$Area\ \ \Delta XCA = \frac{ar}{a+ar}\cdot 1 = \frac{r}{1+r}.$$



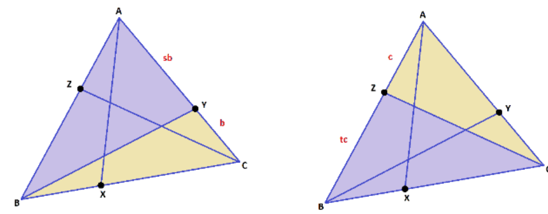
In the same way,

$$Area\ \ \Delta CYB = \frac{1}{1+s}$$

$$Area\ \ \Delta YAB = \frac{s}{1+s}$$

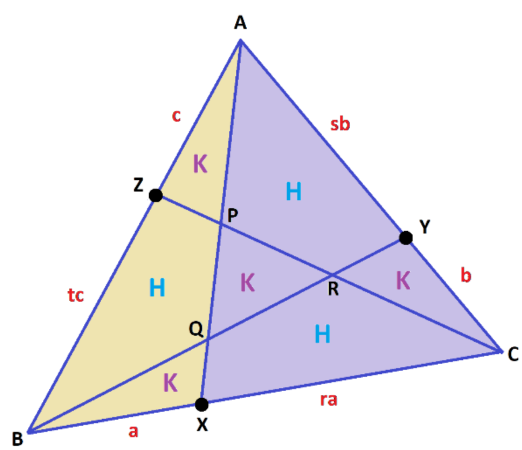
$$Area\ \ \Delta AZC = \frac{1}{1+t}$$

$$Area\ \ \Delta ZBC = \frac{t}{1+t}$$



THE MARVEL OF THE OPENING PUZZLE

Suppose the four interior triangles each have area K and the three interior quadrilaterals each have area H.



We see then that

$$2K + H = \frac{1}{1+r}$$

$$2K + 2H = \frac{r}{1+r}$$

Subtracting gives

$$H = \frac{r-1}{1+r}$$

and then solving for K gives

$$K = \frac{2-r}{2+2r}.$$

Similarly we can argue that

$$H = \frac{s-1}{1+s}$$

and that

$$H = \frac{t-1}{1+t}.$$

From $\frac{s-1}{1+s} = \frac{r-1}{1+r}$ and from $\frac{t-1}{1+t} = \frac{r-1}{1+r}$, it then follows that $r = s = t$.

For the opening puzzler, the three Cevians meet the opposite sides in the same ratio.

But more is true since we have not yet used the fact that the innermost triangle has the same area K as well.

Routh's theorem, with $r = s = t$, gives

$$K = \frac{\left(r^3-1\right)^2}{\left(1+r+r^2\right)^3} = \frac{\left(r-1\right)^2\left(1+r+r^2\right)^2}{\left(1+r+r^2\right)^3}. = \frac{\left(r-1\right)^2}{1+r+r^2}$$

But we've also seen $K = \frac{2-r}{2+2r}$. Equating produces

$$r^2 = r+1.$$

Thus $r = \frac{1+\sqrt{5}}{2}$, the Golden Ratio.

Surprise!

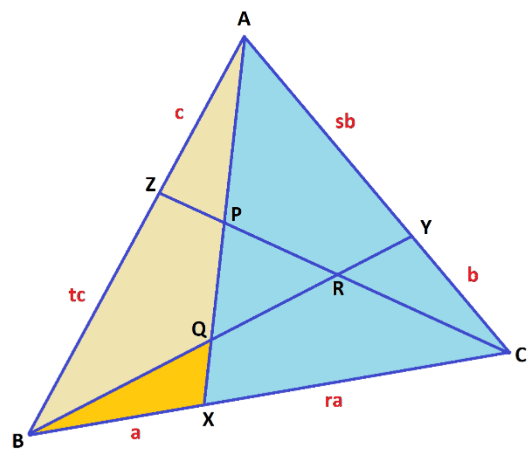
For the opening puzzler, the three Cevians each meet the opposite sides in the Golden Ratio.

A LOGIC TRAP!

We can't stop our discussions here. All we have proved is that *IF* there is a triangle that can be divided by three Cevians into four triangles of matching areas and three quadrilaterals of matching areas, then each Cevian meets a side of the triangle in the Golden Ratio. But maybe there is no triangle that possesses three Cevians that work his way! This entire discussion could be meaningless.

We need to do some more work.

The same February 2016 essay gives a formula for the area of triangle ABQ.



And as we know the area of triangle ABX is $\frac{1}{1+r}$, we get that

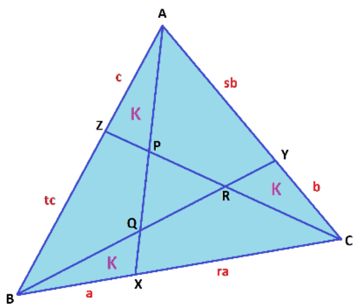
$$Area\ \Delta AXQ = \frac{1}{1+r} - \frac{s}{1+s+sr} = \frac{1}{(r+1)(1+s+sr)}.$$

Similarly,

$$Area\ \Delta CYR = \frac{1}{(s+1)(1+t+ts)}$$

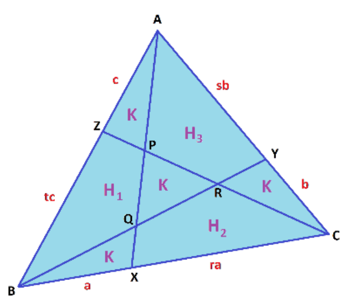
$$Area\ \Delta AZP = \frac{1}{(t+1)(1+r+rs)}.$$

We see then that if $r = s = t$, then the three areas labeled K are equal. Their common value is $\dfrac{1}{(r+1)(1+r+r^2)}$.



As we have seen, Routh's theorem gives $Area\,\Delta PQR$ foe the situation with $r = s = t$ as $\dfrac{(1-r)^2}{1+r+r^2}$. For this to equal $\dfrac{1}{(r+1)(1+r+r^2)}$ one checks we need $r^2 = r+1$, that is, r to be the Golden Ratio.

We have: *If the three Cevians of a triangle each intercept the side of a triangle in the Golden Ratio, then each triangle labeled K has the same area.*



But does this mean the three quadrilaterals have the same area?

Following the labeling in the diagram, and noting that we're assuming $r = s = t$ (with common value the Golden Ratio) we have

$$Area\,H_1 = \frac{1}{r+1} - 2K$$
$$Area\,H_2 = \frac{1}{r+1} - 2K$$
$$Area\,H_3 = \frac{1}{r+1} - 2K.$$

Thus the three quadrilaterals have the same area too in this scenario.

In summary

> For any triangle, the three Cevians that each divide the sides of the triangle in the Golden Ratio do indeed subdivide the triangle into seven regions: four triangles of the same area and three quadrilaterals of the same area.

This tightens our logic. We have now established that three Cevians of a triangle of a triangle create four triangles of equal area and three quadrilaterals of equal area precisely when each Cevian intercepts the side it meets in the Golden Ratio.

Phew! We have established something meaningful in this essay!

Comment: I do not know of the origin of this result. My colleague Izán Péraz recently asked me to solve the opening puzzle but was not sure where he first encountered it.

RESEARCH CORNER

Following the notation of this essay, might there be three non-identical values for r, s, and t for some triangle which give four inner triangles of the same area? Might there instead be three non-identical values for r, s, and t which give three quadrilaterals of the same area?

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