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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JANUARY 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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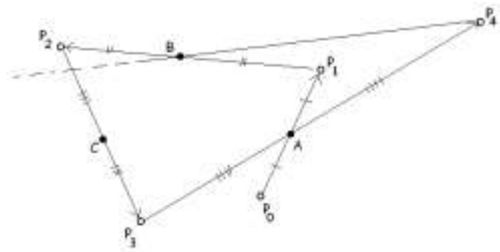
I am learning that a number of schools are adopting the "story of symmetry" approach to quadratics a la www.gdaymath.com/courses/quadratics, promoting "nutting things out" over rote doing and memorization. Check it out.

Let's explore one of my favorite puzzles and push it new directions!

THE CLASSIC LEAPFROG PUZZLE:
Three points A, B, and C are fixed in the plane. A frog, starting at a position P_0, jumps in a straight line towards and over the point A to land at a position P_1 with distances P_0A and AP_1 matching. (Thus A is the midpoint of $\overline{P_0P_1}$.)



The frog then jumps in a straight line towards and over B to land at position

P_2 with distances P_1B and BP_2 matching. And then again towards and over point C, and then towards and over point A, then B, then C, then A, then B, and so on, to create a sequence of landing positions $P_0, P_1, P_2, P_3, P_4, \ldots$ with points A, B, and C cycling as midpoints of the consecutive segments $\overline{P_iP_{i+1}}$.

Prove that the frog is sure to return to start after a finite number of jumps!

VARIATION 1: *The points A, B, C, and P_0 need not be in the same plane! Show that the frog is sure to return to start even if it is leaping in three-dimensional space.*

VARIATION 2: Leap-frogging over a point is the same, geometrically, as rotating 180° about that point. Let's change the measure of that angle of rotation!

Three points A, B, and C are fixed in the plane. Rotate a point P_0 about the point A counter-clockwise through an angle of 60°. Let P_1 be its image point. Now rotate P_1 60° counter-clockwise about the point B to land at P_2, and then rotate P_2 60° counterclockwise about the point C to land at P_3. Keep repeating this process, rotating 60° about A, B, and C in turn.

Show that $P_6 = P_0$. (That is, this process is sure to return to start in six moves!)

VARIATION 3: A variation of variation 2.

Two points A and B are fixed in the plane. Rotate a point P_0 about A counter-clockwise through an angle of 90°. Let P_1 be its image point. Now rotate P_1 a quarter turn counter-clockwise about B to land at P_2. Keep repeating this process, rotating 90° alternately about A, and B.

Show that $P_4 = P_0$.

VARIATION 4: Leap-frogging over a point A has the feel of a reflection, even though it is not one: we land the same distance away from A but on the opposite side of A. Let's turn the original problem into one about reflections.

Suppose L_A, L_B, L_C are three lines in a plane passing through a common point. Choose any point P_0 in the plane and let P_1 be its reflection across L_A. Let P_2 be the reflection of P_1 across L_B, and P_3 the reflection of P_2 across L_C. Continue this process, creating a sequence of points $P_0, P_1, P_2, P_3, P_4, \ldots$ by repeatedly reflecting across each of L_A, L_B, and L_C in turn.

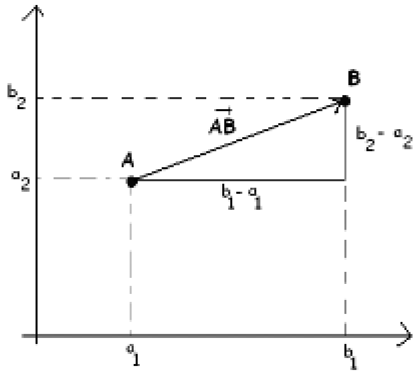
Show that $P_6 = P_0$.

VARIATION 5:
Repeat the previous puzzle, but this time assume that L_A, L_B, L_C are three parallel lines.

SHIFTING POINTS

If I am at the point $(3,7)$ and wish to move to the point $(5,20)$, I need to shift two units horizontally and thirteen units vertically.

More generally, the shift from $A = (a_1, a_2)$ to $B = (b_1, b_2)$, which I'll denote as $\overrightarrow{AB}$, is given by a change of $b_1 - a_1$ units horizontally and $b_2 - a_2$ units vertically.



To be painfully explicit, adding a change of $b_1 - a_1$ to the x-coordinate of A and a change of $b_2 - a_2$ to the y-coordinate of A should give us the point B. And it does.

$$\left(a_1 + \left(b_1 - a_1\right), a_2 + \left(b_2 - a_2\right)\right) = \left(b_1, b_2\right) = B$$
.

This formula reads:

The x-component of A plus the difference of the x-components of B and A equals the x-component of B. Ditto for the y-components.

This is tedious to spell out. To greatly simplify matters, let's write a shift as a difference of two points:

$$\overrightarrow{AB} = B - A.$$

Our formula then reads:

$$A + \overrightarrow{AB} = B$$

(which is equivalent to $A + \left(B - A\right) = B$.)

In classical geometry it does not make sense to add and subtract points. But these formulas are valid and correct when interpreted at the level of components in coordinate geometry.

To test our work, starting at A and shifting only halfway from A to B should land as at the midpoint of $\overline{AB}$. Let' see if it does:

$$A + \frac{1}{2}\overrightarrow{AB} = A + \frac{1}{2}\left(B - A\right) = \frac{A}{2} + \frac{B}{2}.$$

Reading this at the component level, $\frac{A}{2} + \frac{B}{2}$ has x-coordinate half the x-coordinate of A plus half the x-coordinate of B, which is $\frac{a_1 + b_1}{2}$. Ditto for the y-coordinates. Thus $A + \frac{1}{2}\overrightarrow{AB}$ is the point $\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right)$, which is indeed the midpoint!

Comment: This algebraic thinking about points is beautiful. Here's the five-sevenths point formula in case you ever need it!

$$A + \frac{5}{7}\overrightarrow{AB} = A + \frac{5}{7}\left(B - A\right)$$
$$= \left(\frac{2}{7}a_1 + \frac{5}{7}b_1, \frac{2}{7}a_2 + \frac{5}{7}b_2\right).$$

SOMETHING COOL: Let A, B, C be the three vertices of a triangle.

Start at the midpoint of $\overline{AB}$ and walk one third of the way toward C. This lands you at:

$$\frac{A+B}{2} + \frac{1}{3}\left(C - \frac{A+B}{2}\right) = \frac{A+B+C}{3}.$$

So too does starting at the midpoint of $\overline{BC}$ and walking one third of the way towards A, and starting at the midpoint of $\overline{AC}$ and walking one third of the way towards B. (Check this!)

This proves that the three medians of a triangle are concurrent, that they meet at their one-third points, and that the coordinates of the centroid (the point of concurrency) are given by $\frac{A+B+C}{3}$, the average of the x-coordinates of the three vertices and the average of their three y-coordinates!

Comment: This result holds as is for triangles sitting in three-dimensional space too!

SOLVING THE LEAPFROG PUZZLE:

Suppose A, B, and C are three points in the plane. If a leaping frog starts at P_0 and leaps over A, it will land at:

$$\begin{aligned} P_1 &= P_0 + 2\overrightarrow{P_0A} \\ &= P_0 + 2(A - P_0) \\ &= 2A - P_0 . \end{aligned}$$

(This formula reads: "twice the point we're leaping over minus the point we just leapt from.)

Jumping over B then lands the frog at:

$$\begin{aligned} P_2 &= 2B - P_1 \\ &= 2B - 2A + P_0 . \end{aligned}$$

Jumping over C:

$$P_3 = 2C - 2B + 2A - P_0 .$$

Jumping over A:

$$\begin{aligned} P_4 &= 2A - 2C + 2B - 2A + P_0 \\ &= -2C + 2B + P_0 . \end{aligned}$$

Jumping over B:

$$\begin{aligned} P_5 &= 2B + 2C - 2B - P_0 \\ &= 2C - P_0 . \end{aligned}$$

Jumping over C:

$$\begin{aligned} P_6 &= 2C - 2C + P_0 \\ &= P_6 \end{aligned}$$

which, like magic, is back to start!

SOLVING VARIAION 1:

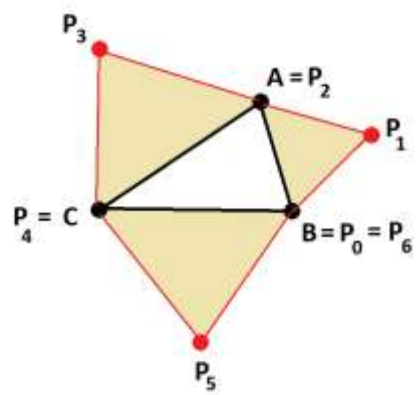
Our work with shifts applies to points in <u>any</u> dimension! The leap-frog puzzle and its proof hold up just fine in three dimensions.

Challenge: Can you draw a three-dimensional picture illustrating the return of the frog after six leaps?

SOLVING VARIATION 2:

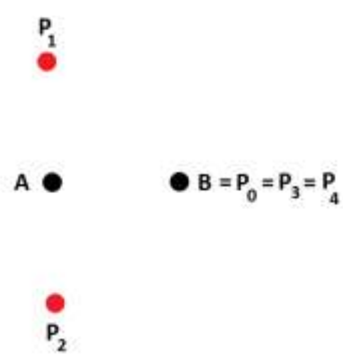
After each rotation of 60° our sense of orientation is altered by 60°. After six such moves, our orientation is returned to its original sense. Thus the combined effect of the six moves in this puzzle is a geometric transformation of the plane that does not change orientation. It must be a translation.

Twitter follower @daveinstpaul points out that is easy to see what this translation is: the point B does not move so it must be the zero translation. Thus all points are returned to their original locations after the six moves.



SOLVING VARIATION 3:

Following the same reasoning as above, the composition of four 90° rotations is a translation. Also, choose $P_0 = B$ shows that this translation is the zero translation.



PLANAR POINTS AS COMPLEX NUMBERS:
There is a natural way to regard points in the coordinate plane as complex numbers in the complex plane:

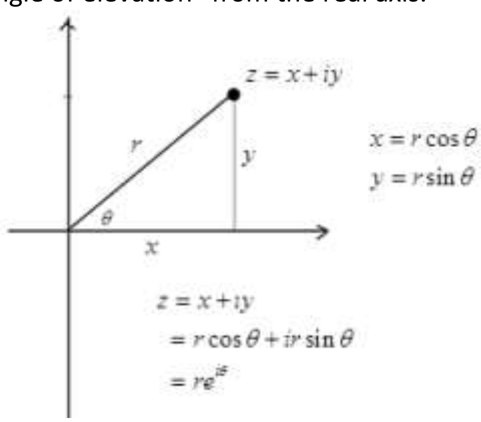
$$(\mathrm{x},\mathrm{y}) \leftrightarrow x+iy.$$

The x - and y -coordinates of a point match the real and imaginary parts of the associated complex number.

Comment: In classical geometry one cannot add and subtract points: we had to view our formulas as statements valid at the component-level. But it is legitimate to add and subtract complex numbers. Moreover, our previous component-level thinking translates into correct arithmetic at the level of the real and imaginary parts of complex numbers.

By viewing points as complex numbers we can bring in our full knowledge of complex arithmetic and use it to advance our study of geometry.

In particular, recall that every complex number can be written in the form $re^{i\theta}$. Here r is its distance from the origin in the complex plane and θ is its argument, its "angle of elevation" from the real axis.



Comment: Using Euler's famous formula $e^{i\theta} = \cos\theta + i\sin\theta$ obviates battling with the trigonometry directly.

From now on we shall use the words *point* and *complex number* interchangeably: every complex number represents a point in the plane, and vice versa.

Rotations and Reflections:
To rotate a point, that is, a complex number, $z = re^{i\theta}$ through an angle of measure α counter-clockwise about the origin all we need do is increase its argument by α. We can accomplish this by multiplying the complex number by $e^{i\alpha}$:

$$z = re^{i\theta} \to e^{i\alpha}z = re^{i(\theta+\alpha)}.$$

To rotate z about a point a different from the origin through this angle, do the following:

Translate z and a so that a lands at the origin. Rotate the translate of z about the origin. Translate the image point back.

$$\begin{aligned} z &\to z-a \\ &\to e^{i\alpha}(z-a) \\ &\to e^{i\alpha}(z-a)+a = e^{i\alpha}z - e^{i\alpha}a + a. \end{aligned}$$

.

Complex conjugation $z \to \overline{z}$ reflects complex numbers across the real axis. To reflect a complex number z about a line through the origin with angle of elevation α do the following:

Rotate z and the line about the origin through an angle of $-\alpha$. (The image of the line is the real axis.) Reflect the rotated copy of z about the real axis. Rotate its image back through an angle α.

$$\begin{aligned} z &\to e^{-i\alpha}z \\ &\to \overline{e^{-i\alpha}z} = e^{i\alpha}\overline{z} \\ &\to e^{i\alpha}e^{i\alpha}\overline{z} = e^{2i\alpha}\overline{z}. \end{aligned}$$

ALTERNATIVE SOLUTION TO VARIATION 2:
Let a, b, and c be the three complex numbers representing the points A, B, and C, and let $w = e^{i\frac{\pi}{3}}$. Multiplication by w affects a 60° counter-clockwise rotation about the origin.

Let p_0 be our starting point. Then the moves of variation 2 give:

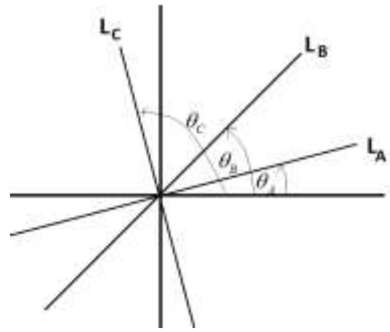
$$\begin{aligned}
p_1 &= wp_o - wa + a \\
p_2 &= w^2 p_0 - w^2 a + wa - wb + b \\
p_3 &= w^3 p_0 - w^3 a + w^2 a - w^2 b + wb - wc + c \\
p_4 &= w^4 p_0 - w^4 a + w^3 a - w^3 b + w^2 b - w^2 c + wc \\
&\quad - wa + a \\
p_5 &= w^5 p_0 - w^5 a + w^4 a - w^4 b + w^3 b - w^3 c + w^2 c \\
&\quad - w^2 a + wa - wb + b \\
p_6 &= w^6 p_0 - w^6 a + w^5 a - w^5 b + w^4 b - w^4 c + w^3 c \\
&\quad - w^3 a + w^2 a - w^2 b + wb - wc + c \\
&= w^6 p_0 + (1 + w^3)(stuff)
\end{aligned}$$

Now $w^3 = \left(e^{i\frac{\pi}{3}}\right)^3 = e^{i\pi} = -1$ and $w^6 = (-1)^2 = 1$. It follows that:

$$p_6 = 1 \cdot p_0 + 0 = p_0.$$

> **Exercise:** Mimic this proof to give an alternative solution to variation 3.

SOLVING VARIATION 4:
We can assume that all three lines meet at the origin. Suppose line L_A makes angle θ_A with the real axis, line L_B angle θ_B, and line L_C angle θ_C.



Let p_o be our starting position in the complex plane. Then:

$$\begin{aligned}
p_1 &= e^{2i\theta_A}\overline{p_o} \\
p_2 &= e^{2i\theta_B}e^{-2i\theta_A}p_o \\
p_3 &= e^{2i\theta_C}e^{-2i\theta_B}e^{2i\theta_A}\overline{p_o} \\
p_4 &= e^{2i\theta_A}e^{-2i\theta_C}e^{2i\theta_B}e^{-2i\theta_A}p_o = e^{-2i\theta_C}e^{2i\theta_B}p_o \\
p_5 &= e^{2i\theta_B}e^{2i\theta_C}e^{-2i\theta_B}\overline{p_o} = e^{2i\theta_C}\overline{p_o} \\
p_6 &= e^{2i\theta_C}e^{-2i\theta_C}p_o = p_0
\end{aligned}$$

SOLVING VARIATION 5:
Draw a line through P_0 perpendicular to the three parallel lines. Suppose this line meets L_A, L_B, and L_C at points A, B, and C, respectively. Then playing variation 5 is identical to playing the original leapfrog puzzle with the points A, B, and C. We must return to P_0 after six moves.

RESEARCH CORNER

Lots of questions!

1. The original leapfrog puzzle has us leaping over three points in turn. Is the count of three special?
   a) Show, with five points, we are sure to return to start in ten moves.
   b) Is it possible to ever return to start alternately leapfrogging over two points?
   c) For which N is leapfrogging over N points sure to eventually have us return to start?

2. Can we change the count of lines in variation 4?

3. Do the lines need to be concurrent in variation 4?

4. Can you develop other puzzles based on the ideas of variations 2 and 3? That is, can you change the number of points and the measures of the rotations about those points to create other compositions of rotations that reduce to the zero translation?

5. The original leap-frog puzzle works in three dimensional space. Does variation 4 work in three dimensions too? (To reflect over a line in three-dimensional space, reflect in the plane defined by the line and the point you are moving from. In other words, rotate the point 180° about the line!)

6. Consider three planes in space, no two of which are parallel. If I reflect over the first plane, then the second, then the third, and then the first, second, and third each one more time, will I be back to start?

7. The angle bisectors of a triangle are concurrent and reflections across those angle bisectors map the sides of the triangle amongst themselves. *Is every set of three concurrent lines a set of angle bisectors for some triangle?* If so, we can develop a much simpler and direct geometric proof of variation 4 that avoids complex arithmetic. (Can you see how?)

8. If we say that parallel lines do meet, at some "point at infinity," then variation 5 is just a specific instance of variation 4. Further, in this world of "projective geometry" in which every two lines do meet at a point and every two points do "meet" at a line, the words "point" and "line" are interchangeable, and any result that is true about points and lines is also true if restated about lines and points.

I, personally, can't shake the feeling that the original leapfrog puzzle and variation 4 are just dual versions of each other. (Look too at their proofs.)

Is there a best philosophical way to think about these puzzles as part of one same idea?

tanton.math@gmail.com