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CURIOUS MATHEMATICS FOR FUN AND JOY

FEBRUARY 2021

THIS MONTHS' PUZZLER:

The six square numbers

$$1,\ 49,\ 289,\ 961,\ 2401,\ 5041$$

(which are 1^2, 7^2, 17^2, 31^2, 49^2, and 71^2) have the property that the average of any two consecutive terms is another square number.

$$Ave(1,49) = 25 = 5^2$$

$$Ave(49,289) = 169 = 13^2$$

$$Ave(289,961) = 625 = 25^2$$

$$Ave(961,2401) = 1681 = 41^2$$

$$Ave(2401,5041) = 3721 = 61^2$$

$$\ldots$$

Can a seventh number be added to the list? An eighth?

Can the list be extended indefinitely to create an increasing sequence of square numbers with conseuctive averages square?

PYTHAGOREAN TRIPLES

Suppose A^2, B^2, and C^2 are three square numbers with

$$\frac{A^2+B^2}{2}=C^2.$$

Then setting

$$x=\frac{|A-B|}{2},\ y=\frac{A+B}{2}, \text{ and } z=C$$

gives three values satisfying $x^2+y^2=z^2$. (Check this.)

Are x and y sure to be integers?

If A and B are both even, then yes.

Challenge 1: *If A and B are both even, show that C must be as well.*

If A and B are both odd, then $A-B$ and $A+B$ are both even, and again the answer is yes.

Challenge 2: *If A and B are both odd, show that C must be odd as well.*

And is not possible to have A odd and B even, or vice versa, since $\left(A^2+B^2\right)/2$ is an integer, namely, C^2.

We have

Three square numbers A^2, B^2, and C^2 satisfying

$$\frac{A^2+B^2}{2}=C^2$$

give a Pythagorean triple $x^2+y^2=z^2$ with x, y, and z as above.

The converse also true.

If x, y, z represent a Pythagorean triple, the algebra shows that $A=|x-y|$, $B=x+y$, $C=z$ are three integers satisfying $\frac{A^2+B^2}{2}=C^2$.

From the triple $(3,4,5)$ we get

$$\frac{1^2+7^2}{2}=5^2.$$

From $(5,12,13)$ we get $\frac{7^2+17^2}{2}=13^2$.

From $(48,55,73)$ we get $\frac{7^2+103^2}{2}=73^2$.

And so on.

From the triple $\left(2mn, m^2-n^2, m^2+n^2\right)$, for any two counting numbers m and n with $m>n$, we get

$$\frac{\left(m^2-n^2-2mn\right)^2+\left(m^2-n^2+2mn\right)^2}{2}=\left(m^2+n^2\right)^2.$$

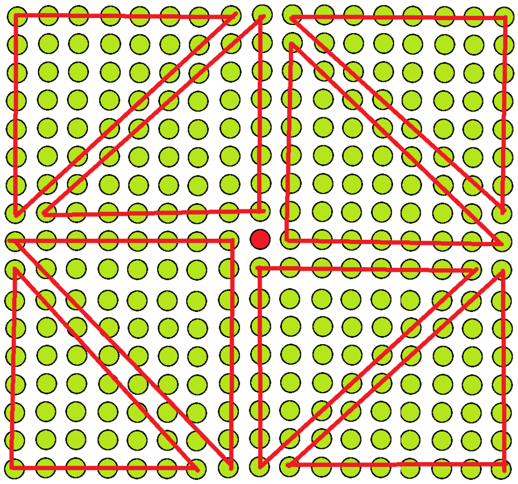
TRIANGULAR AVERAGES

Suppose A, B, and C are three integers satisfying $\frac{A^2+B^2}{2}=C^2$. If these integers are all even, let's delete any common factors of two so that at least one of these values is odd.

Challenge 3: *Show that if one of A, B, or C is odd, then all three values are odd.* (This extends challenge 2.)

So, let's assume our work from now on is solely with odd square numbers.

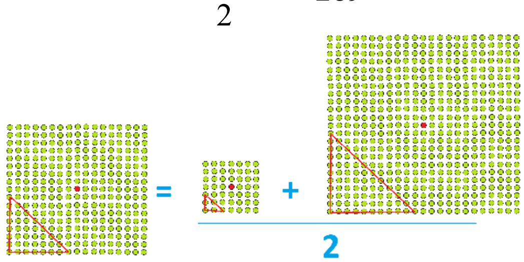
Every odd square number is one more than eight times a triangular number.



$$13^2 = 8(21)+1$$

Thus, two odd squares whose average is another (necessarily) odd square, gives two triangular numbers whose average is again triangular. For instance, from

$$\frac{49+529}{2} = 289$$



we get

$$\frac{(8\cdot 6+1)+(8\cdot 66+1)}{2} = 8\cdot 36+1.$$

That is,

$$8\left(\frac{6+66}{2}\right)+1 = 8\cdot 36+1$$

yielding

$$\frac{6+66}{2} = 36.$$

The average of the third and eleventh triangular numbers is the eighth triangular number.

Challenge 4: *Show that each primitive Pythagorean triple is sure give three* ***odd*** *square numbers with one the average value of the other two, and thus three triangular numbers with one the average value of the other two.*

Can one reverse this process?

SEQUENCE OF SQUARE NUMBERS

Recall that from the Pythagorean triple $\left(2mn, m^2-n^2, m^2+n^2\right)$ for two counting numbers $m>n$, we get

$$\frac{\left(m^2-n^2-2mn\right)^2+\left(m^2-n^2+2mn\right)^2}{2} = \left(m^2+n^2\right)^2.$$

Here the first square number in the numerator is smaller than the second.

Set $n=p$ and $m=p+k$ for some positive values p and k. Then we have

$$\frac{\left((p+k)^2-p^2-2p(p+k)\right)^2+\left((p+k)^2-p^2+2p(p+k)\right)^2}{2}$$
$$=\left((p+k)^2+p^2\right)^2$$

For $n=p+k$ and $m=p+2k$ we get

$$\frac{\left((p+2k)^2-(p+k)^2-2(p+k)(p+2k)\right)^2+\left((p+2k)^2-(p+k)^2+2(p+k)(p+2k)\right)}{2}$$
$$=\left((p+2k)^2+(p+k)^2\right)^2$$

Tedious algebra shows that $(p+k)^2-p^2+2p(p+k)$ and $(p+2k)^2-(p+k)^2-2(p+k)(p+2k)$ differ only by a minus sign, and so, when each is squared, they have the same squared value.

Ooh!

From the Pythagorean triple

$$\left(2mn, m^2 - n^2, m^2 + n^2\right)$$

setting

$n = p$ and $m = p + k$

and then

$n = p + k$ and $m = p + 2k$

gives us square numbers A^2, B^2, C^2 with

$$\frac{A^2 + B^2}{2} \text{ and } \frac{B^2 + C^2}{2}$$

each square numbers.

Setting $n = p + 2k$ and $m = p + 3k$ then gives a square number D^2 such that

$$\frac{C^2 + D^2}{2}$$

is a square number.

We can continue this indefinitely.

Example: Setting $p = 1$ and $k = 1$ so that we choose, in turn,

$$n = 1,\ m = 2$$
$$n = 2,\ m = 3$$
$$n = 3,\ m = 4$$
$$\ldots$$

yields the sequence of square numbers

$$1,\ 49,\ 289,\ 961,\ 2401,\ 5041,\ 9409,\ 16129, \ldots$$

with each pair of consecutive terms averaging to a square number.

This yields the sequence of triangular numbers $0, 6, 36, 120, 300, 630, 1176, \ldots$ with consecutive terms having average value a triangular number.

Example: Setting $p = 1$ and $k = 3$ so that we choose, in turn,

$$n = 1,\ m = 4$$
$$n = 4,\ m = 7$$
$$n = 7,\ m = 10$$
$$\ldots$$

yields the sequence of square numbers

$$49,\ 529,\ 7921,\ 36481,\ \ldots$$

with each pair of consecutive terms averaging to a square number.

RESEARCH CORNER

Can we allow n and m to be negative integers? What pairs of square numbers arise from

$$\ldots$$
$$n = -2,\ m = -1$$
$$n = -1,\ m = 0$$
$$n = 0,\ m = 1$$
$$n = 1,\ m = 2$$
$$n = 2,\ m = 3$$
$$n = 3,\ m = 4$$
$$\ldots$$

and other such lists?

Does every increasing sequence of square numbers with the property that each pair of consecutive terms averages to a square number arise in the manner of this essay? Have we classified them all?

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