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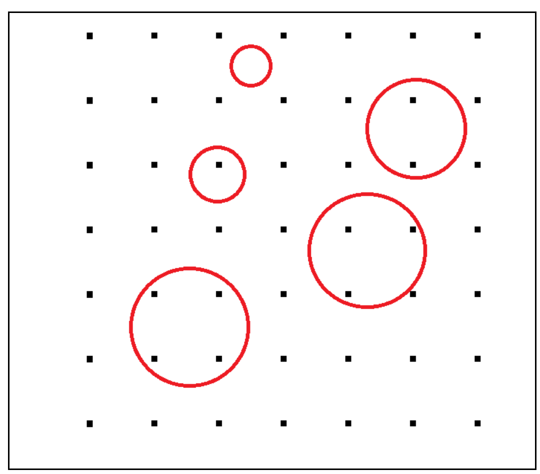
CURIOUS MATHEMATICS FOR FUN AND JOY

FEBRUARY 2020

THIS MONTHS' PUZZLER:

Consider the square lattice of points in the plane with integer coordinates.

For each counting number $N \geq 0$ is there a circle in the plane with N lattice points in its interior (and none on its boundary)?



SPECIAL POINTS IN THE PLANE

There are special points P in the plane such that any two distinct lattice points with integer coordinates have different distances from P.

For instance, $P=\left(\sqrt{2},\sqrt{3}\right)$ is such a point.

To see this, suppose to the contrary that (a,b) and (c,d) are points with integer coordinates the same distance from P. Then, according to the distance formula, squared, we must have

$$\left(a-\sqrt{2}\right)^2+\left(b-\sqrt{3}\right)^2 =\left(c-\sqrt{2}\right)^2+\left(d-\sqrt{3}\right)^2$$

or, equivalently,

$$2\sqrt{2}(c-a)+2\sqrt{3}(d-b)=c^2+d^2-a^2-b^2$$

Squaring gives

$$8\sqrt{6}(c-a)(d-b)=\left(c^2+d^2-a^2-b^2\right)^2 -8(c-a)^2-12(d-b)^2$$

The right side of this equation is an integer. Since $\sqrt{6}$ is not a ratio of two integers (it is an irrational number), we must have that either $c-a=0$ or $d-b=0$ (and the right side is zero as well). But from our very first equation, $a=c$ implies that $\left(b-\sqrt{3}\right)^2=\left(d-\sqrt{3}\right)^2$, forcing $b=d$ as well. (Think through this. This relies in knowing that $\sqrt{3}$ is irrational too.)

Similarly, the assumption that $b=d$ forces $a=c$ as well.

Thus for lattice points (a,b) and (c,d) to be equidistant from $P=\left(\sqrt{2},\sqrt{3}\right)$, we must have $a=c$ and $b=d$. That is, they are the same point.

Challenge: a) Show that the point $P=\left(\sqrt{2},\frac{1}{3}\right)$ has this property too. That is, show that any two lattice points with integer coordinates have distinct distances from P.

b) Is there a point with both coordinates rational such that no two lattice points have the same distance from this point?

Advanced Comment: One can argue that almost <u>all</u> points in the plane have this desired property! Assuming some advanced undergraduate measure theory we can argue this way:

- There are countably many lattice points in the plane.
- There are countably many pairs of lattice points, and thus countably many lines of equidistance in the plane between pairs of lattice points.
- Each line in the plane has measure zero and a countable collection of lines thus has measure zero as well.
- The plane is a set of infinite measure.
- Consequently, almost all points in the plane do not lie on a line of equidistance between any two lattice points.

Thus, for almost all points in the plane, the distances between it and each lattice point are distinct.

Challenge: Prove, in three-dimensional space, that the no two points with all three coordinates an integer are equidistant from the point $\left(\sqrt{2},\sqrt{3},\frac{1}{3}\right)$.

Returning focus to the point $P=\left(\sqrt{2},\sqrt{3}\right)$ let's order the lattice points in the plane Q_1, Q_2, Q_3, Q_4,... with Q_1 being the lattice point closest to P, Q_2 the lattice point second closest to P, and so on. Let d_1, d_2, d_3, d_4, ... be the (distinct) distances of these lattice points from P. Also, let r_0 be a number between 0 and d_1, r_1 be a number between d_1 and d_2, r_2 be a number between d_2 and d_3 , and so on.

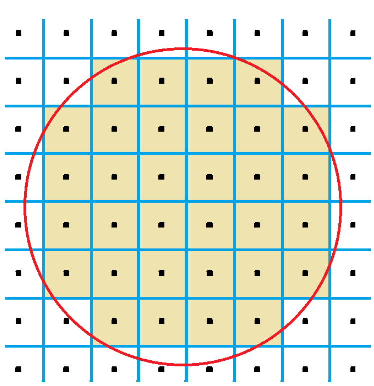
For each counting number $N \geq 0$, the circle with center P and radius r_N has precisely N lattice points in its interior (and no lattice points on its boundary).

This answers the opening puzzler to the affirmative.

ESTIMATING RADII

Draw a unit square around each lattice point with the lattice point at its center.

If a circle in the plane has N interior lattice points, then its area can be approximated as the sum of N unit square areas.



Thus the radius r of a circle containing precisely N interior lattice points satisfies the approximate equation $\pi r^2 \approx N$ and this gives

$$r \approx \sqrt{\frac{N}{\pi}}$$

and if r is large, this approximation should be fairly good.

[If this argument feels too "loose" to you, we can argue as follows. Suppose a circle with center P and radius r contains N interior lattice points. Let S be the region composed of all unit squares with centers lattices point inside this circle.

Since a unit square has maximal "width" $\sqrt{2}$ —the length of its diagonal—the set S lies entirely within the circle of the same center P but radius $r+\sqrt{2}$. (Each square in S that "extends" beyond the circle of radius r, extends no more than a radius of $r+\sqrt{2}$.) Also, the set S completely covers the circle with center P and radius $r-\sqrt{2}$. (Any square cell that is not part of S but possesses a portion of area that lies within the circle with center P of radius r lies outside the circle with the same center and radius $r-\sqrt{2}$.) Comparing areas, we have

$$\pi\left(r-\sqrt{2}\right)^2 \leq N \leq \pi\left(r+\sqrt{2}\right)^2.$$

Thus

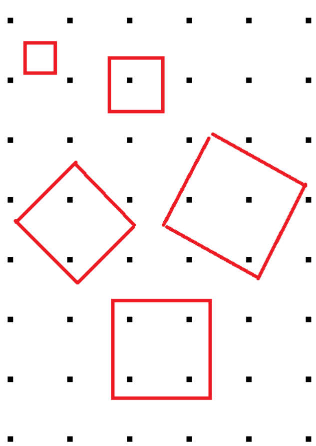
$$\pi\left(1-\frac{\sqrt{2}}{r}\right)^2 \leq \frac{N}{r^2} \leq \pi\left(1+\frac{\sqrt{2}}{r}\right)^2.$$

For large values of r, the left and right terms of this compound inequality are close to the value $\pi \times (1 \pm 0) = \pi$, showing that $\frac{N}{r^2} \approx \pi$, as we claimed.]

Comment: This approach gives a means to estimate the value of π: Draw a circle of some large radius r on a sheet of graph paper and count the number N of grid points that lie within the circle. Then $\pi \approx \frac{N}{r^2}$. (Does this method seem "efficient"?)

RESEARCH CORNER

In the 1950s J. Browkin proved that for each counting number $N \geq 0$ there is a square in the plane that contains in its interior precisely N lattice points (with no lattice points on its boundary).



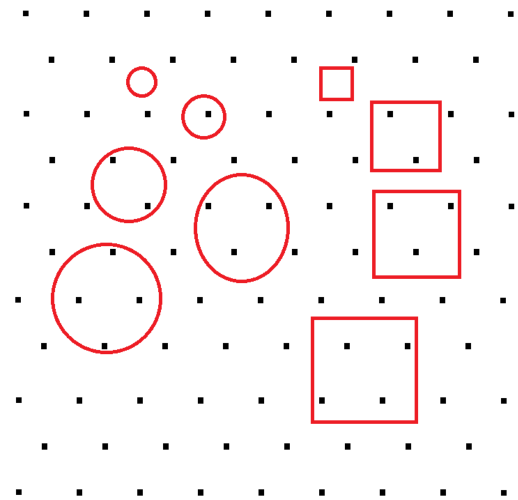
His proof was involved and complicated. Is there a "simple" proof of this fact?

At the very least, are there some values for N for which finding a square with N interior lattice points is straightforward?

If $N = a^2$, a square number, then matters are fine. Numbers of the form $N = a^2 - a$ and $N = a^2 - 4$ are fine too. (Why?)
If $N = a^2 + b^2$, a sum of two square numbers, then both $N-1$ and $N+3$ are good. (Why? Look up Pick's Theorem.)

Are we on a path to proving that every for very natural number N there is a square in the plane containing precisely N lattice points?

Care to consider circles and squares in the plane containing particular counts of equilateral triangular lattice points?



James Tanton
tanton.math@gmail.com