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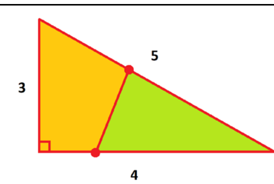
CURIOUS MATHEMATICS FOR FUN AND JOY

FEBRUARY 2019

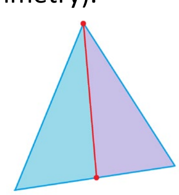
PUZZLE: A SPECIAL DIVIDING LINE

1. Every triangle possesses at least one line through it that simultaneously divides its area and its perimeter each in half. (Why? How can we be sure?) For an isosceles triangle, the line of symmetry of the triangle is such a line.

Describe the exact location of such a dividing line for a $3-4-5$ right triangle.



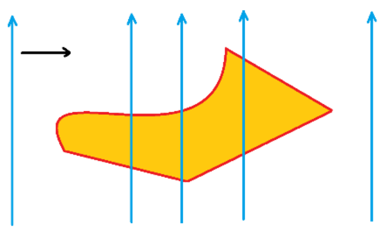
2. If a line from one vertex of a triangle simultaneously divides its area and perimeter each in half, prove that the triangle must be isosceles (with this line its line of symmetry).



DIVIDING LINES IN FIGURES

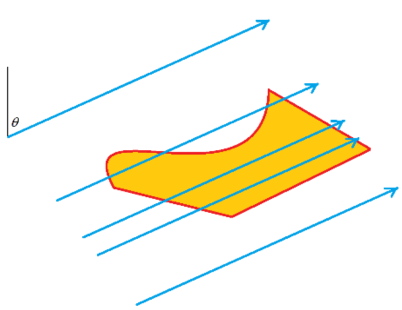
There is nothing special about triangles in the opening puzzle: for any standard figure in the plane of finite area and finite perimeter (let's avoid strange, disconnected, fractally things!) there exists a line that simultaneously divides its area and its perimeter in half.

To see why, imagine sweeping a directed line across the figure.



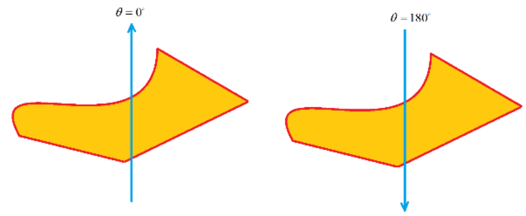
In the diagram above we start with a vertical directed line to left of the figure and sweep it rightwards to the right of the figure. Initially 100% of the figure's area is to the right of the directed line and we end with 0% of the figure's area to its right. There must be an intermediate position at which 50% of the figure's area lies to the right of the directed line. That is, there exists a vertical line that divides the area of the figure in half.

The directed line need not be vertical here. *For any angle θ from the vertical there exists a directed line at that angle that divides the area of the figure in half.*



Of course, the directed lines that divide the area in half might or might not divide the perimeter in half as well.

Consider again the vertical directed line that divides the area in half.

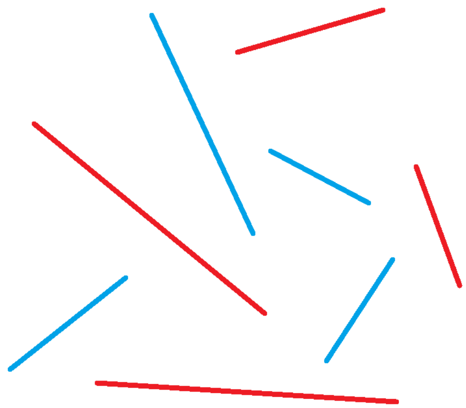


If this line happens to divide the perimeter in half, then we're done! We've found a special line that simultaneously equally splits the area and the perimeter of the figure. If it doesn't, let P_R be the length of perimeter to the right of the directed line and P_L be the length of perimeter to its left. Then $P_R - P_L$ is nonzero.

In fact, for each angle θ, set $f(\theta)$ to be the difference of perimeters to the right and the left of the line at angle θ that divides the area in half. We have $f(0^\circ) = P_R - P_L$. We also have $f(180^\circ) = P_L - P_R$ since the line at 180° is the same as the line at 0°, but of the opposite orientation.

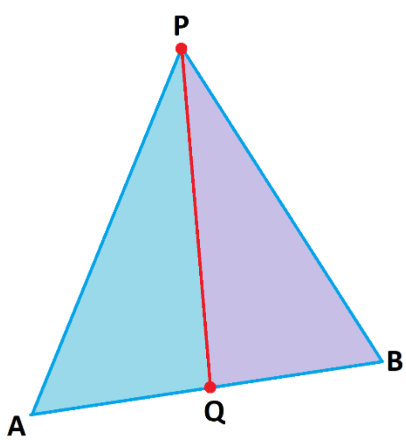
If the figure is indeed straightforward and smooth and connected, then $f(\theta)$ is a continuous function on the angle θ with $f(0^\circ)$ and $f(180^\circ)$ having opposite sign. There must be an intermediate angle θ^* at which $f(\theta^*) = 0$. That is, the line at angle θ^* not only divides the area of the figure in half, but divides the perimeter of the figure in half too.

CHALLENGE: Four red line segments and four blue line segments are drawn on a page. Prove that it is possible to rip the page along a straight-line tear so that the total length of red segments on each section of paper is the same, as is the total length of blue segments on each section.



SPECIAL LINES FORCING ISOSCELES TRIANGLES

Let's look at the second puzzle. Consider a triangle APB and suppose a line through vertex P divides both its area and perimeter in half.



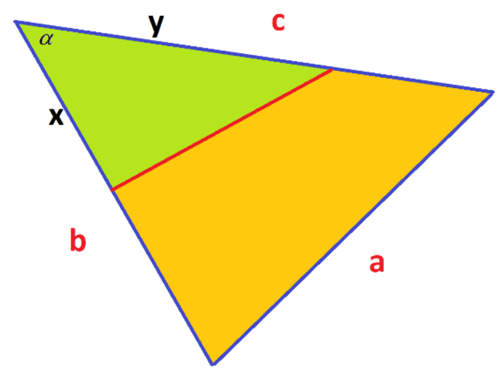
Let Q be the location at which this special line intercepts the side opposite P.

Regarding AQ and BQ as bases, since triangles PAQ and PBQ have the same height and the same area, it follows that $AQ = QB$. That is Q is the midpoint of AB.

Now $PA + AQ$ is half the perimeter of the triangle, as is the sum $PB + BQ$. Since $AQ = BQ$ it follows that $PA = PB$ and the triangle is isosceles.

SPECIAL LINES IN NON-ISOSCELES TRIANGLES

Consider a non-isosceles triangle with side lengths a, b, and c, and suppose a line dividing its area and perimeter in half intercepts the sides of lengths b and c marking off lengths x and y as shown. (By the first section of this essay we know that such a line exists, and by the second section that this line does not pass through a vertex of the triangle.)



Let $P = a + b + c$ be the perimeter of the triangle and $s = \frac{P}{2} = \frac{a+b+c}{2}$ the semi-perimeter. We have

$$x + y = s .$$

The area of the triangle is $\frac{1}{2}bc\sin(\alpha)$ with α the angle between the sides of length b and c. Since $\frac{1}{2}xy\sin(\alpha)$ is half of this, we get

$$xy = \frac{1}{2}bc .$$

This pair of equations gives a quadratic equation

$$x(s-x)=\frac{1}{2}bc$$
$$x^2-sx=-\frac{1}{2}bc$$
$$\left(x-\frac{s}{2}\right)^2=\frac{s^2}{4}-\frac{1}{2}bc$$

yielding

$$x=\frac{s\pm\sqrt{s^2-2bc}}{2}$$
$$y=\frac{s\mp\sqrt{s^2-2bc}}{2}.$$

We need to choose the solutions that give $0<x<b$ and $0<y<c$.

Example: The 3-4-5 right triangle

Here $s=\frac{3+4+5}{2}=6$. But do we work with $a=3$ or $a=4$ or $a=5$?

Trying $a=3$ gives

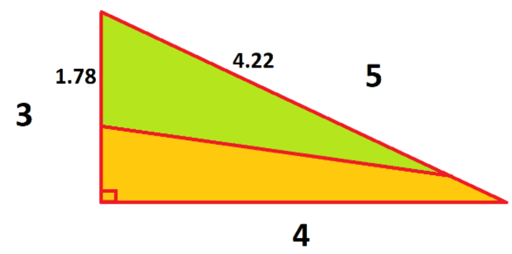
$$x=\frac{6\pm\sqrt{36-40}}{2}$$
$$y=\frac{6\mp\sqrt{36-40}}{2}$$

and we have no solution.

Trying $a=4$ gives

$$x=\frac{6\pm\sqrt{36-30}}{2}=3\pm\frac{1}{2}\sqrt{6}$$
$$y=\frac{6\mp\sqrt{36-30}}{2}=3\mp\frac{1}{2}\sqrt{6}$$

yielding the approximate values 4.22 and 1.78. We have found a line!



Just for completeness, checking $a=5$ gives

$$x=\frac{6\pm\sqrt{36-24}}{2}=3\pm\sqrt{3}$$
$$y=\frac{6\mp\sqrt{36-24}}{2}=3\mp\sqrt{3}$$

yielding the approximate values 4.73 and 1.27 which do not fit within the remaining two side lengths of 3 and 4.

The 3-4-5 right triangle has exactly one special dividing line.

CHALLENGE:
Show that the 7-8-9 triangle has exactly two lines that simultaneously divides its area and perimeter each in half.

Find an isosceles triangle with exactly two lines that simultaneously divides its area and perimeter each in half.

Find a triangle with exactly three such lines.

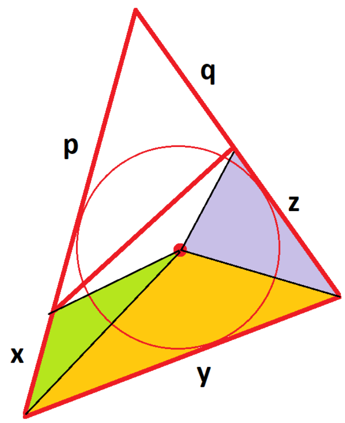
INCENTERS

Every triangle contains an interior circle that is tangent to all three sides of the triangle. The center of this circle is called the *incenter* of the triangle and it is the location at which the three angle bisectors of the triangle meet.

Theorem: *Any line that simultaneously divides the area and the perimeter of the triangle in half passes through the incenter of the triangle.*

Here's why.

Suppose such a line misses the incenter I of the triangle. Suppose we have a diagram as shown with sections of the perimeter labeled. Let r be the radius of the incircle.



The sum of areas of the three shaded triangles is less than the area of the quadrilateral in which they sit, which is half the area of the triangle.

But we have $x+y+z$ equal to $p+q$ (half the perimeter) and the sum of their areas is $\frac{1}{2}xr+\frac{1}{2}yr+\frac{1}{2}zr$. The remaining area of the triangle is $\frac{1}{2}pr+\frac{1}{2}qr$, which is the same value! Thus the sum of areas of the shaded triangles is exactly half the area of the triangle. We have a contradiction!

And a similar contraction is reached if the incenter of the circle lies on the other side of the dividing line.

It must be the case then that the incenter lies on the dividing line.

Comment: We can prove that for any polygon that circumscribes a circle, a line that simultaneously divides is area and perimeter in half must pass through the center of that circle.

RESEARCH CORNER

Is there a scalene triangle with three lines that simultaneously divide its area and perimeter in half?

For which values $0<r<1$ can one find a line through a 3-4-5 right triangle that has fraction r of the area and fraction r of the perimeter to one side of that line?

Care to develop a theory about when "r-lines" do and don't exist in triangles?

Reference: When I first started writing this essay, I was not aware of any literature on this topic. I have since learned of Paul Yiu's piece "Lines simultaneously bisecting the perimeter and area of a triangle" (Global Journal of Advanced Research on Classical and Modern Geometries, Vol 5, 2016, pp 7-14.) It's worth a look! There Yiu classifies all the triangles that possess exactly 1, exactly 2 ,and exactly 3 dividing lines. (Can a triangle possess four dividing lines?)

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