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WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

FEBRUARY 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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I am compelled to start a new series of videos explaining my take on the Common Core State Standards (Mathematics) to answer, from my perspective, a whole host of basic questions about them:

1. What are they? Like, really, what are we actually talking about?
2. What is their desired, ultimate goal?
3. What are these strange "practice standards"? Plain English please!
4. Why can't I seem to help my child with his homework anymore?
5. Are other countries doing this sort of thing too?

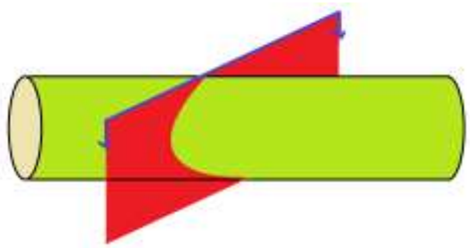
You can find the first of my videos on the front page of www.jamestanton.com. (Or click directly to it at https://www.youtube.com/watch?v=j4I-jkUt49I&feature=youtu.be .)
Thoughts, reactions, and feedback are always most welcome.

WRAPPING-PAPER PUZZLE

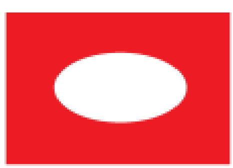
This month's essay is about a paper-slicing puzzle. I'll describe the puzzle here, but it is much more fun and enlightening to watch my goofy video about it. There you can witness the physical beauty of the puzzle.

Here's the video:
www.jamestanton.com/?p=1549
I invite you to watch it now!

WRAPPING-PAPER PUZZLE:
Take a full roll of wrapping paper and slice through it with some cut diagonal to the tube.



The cross section of the slice is oval in shape. (The video shows this.)



The only oval-shaped object we study in high-school mathematics is the ellipse. So here is a first question:

Question 1: *Is the cross-section from slicing a circular cylinder truly an ellipse?*

Two comments about this:

1. We know (or are told) in school geometry that an ellipse arises from slicing a cone, not a cylinder. It is thus not immediately obvious that the shape obtained in this puzzle sits with the family of ellipses. (For instance, if we slice the roll at an extremely oblique angle, say parallel to the tube, the cross section is a region between a pair of parallel lines. No slice of a cone produces a pair of parallel lines.)

2. Because we study very few curved shapes in high-school mathematics it is very natural for folk to think that all oval shapes must be ellipses, just as all U-shaped curves must be parabolas. It would be very constructive if we could help our students discover and prove that the U-shape curve of hanging chain, for example, is not a parabola. (See the opening section of my quadratics course www.gdaymath.com/courses/quadratics. There I show a fun and lovely way to see for yourself that the curve of a hanging chain simply cannot be parabolic.)
I would love our students to realise "Just because something looks like it could be so doesn't actually mean it is so."

There's a second question:

Unroll the wrapping paper from the cut tube. You will see a lovely wavy-shaped border.

(This shows really well in the video!)

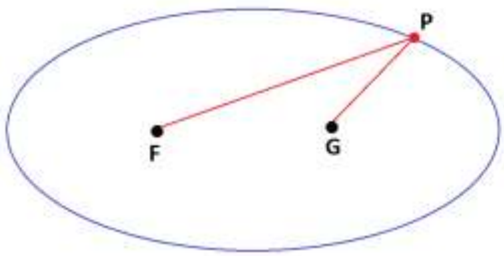
The only wavy curves one studies in standard school mathematics are the trigonometric curves. So …

Question 2: *Is the wavy curve produced by slicing a roll of paper actually a sine curve?*

THE ANSWERS

Question 1:

An ellipse is introduced in high-school mathematics as the set of all points P in a plane whose distances from two fixed points F and G (the *foci*) has a given fixed sum:

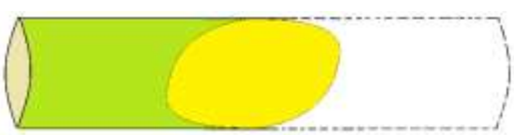


$PF + PG = k$, a fixed value.

One can thus draw an ellipse by taping the ends of a slack length of string at points F and G and pulling the string taut with a pencil. The curve traced swinging the pencil around keeping the string taut is an ellipse.

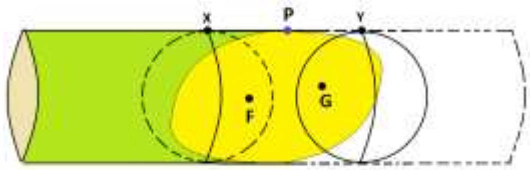
Comment: It is not at all clear that the curves that arise from slicing cones have this fixed distance sum property!

Here is my attempt to draw the oblique cross-section of a cylinder. (I've also outlined the right half of the tube.)



We are wondering if the plane oval shape shown here in yellow is an ellipse.

Imagine pushing a sphere with width matching the width of the cylinder in through the left through the tube until it makes a first point of contact F with the yellow plane cross-section. Do the same for a second sphere pushed in from the right, making the point of contact G in the yellow plane.



The equators on the two spheres shown in the diagram are a constant distance XY apart.

Let P be any boundary point of the yellow region. (We can orient the tube so that the point P is in the top location shown in the diagram.)

Now the line segment $\overline{PF}$ lies in the yellow plane and is a segment tangent to the left sphere. The line segment $\overline{PX}$ is also tangent to the left sphere. By the two-tangents theorem in geometry, these two segments have the same length: $PF = PX$.

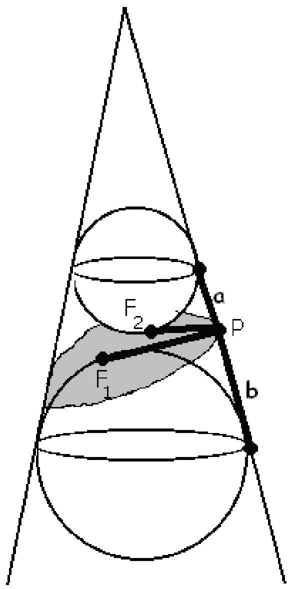
Similarly, the line segments $\overline{PG}$ and $\overline{PY}$ are two tangent segments to the right sphere, and so too are congruent: $PG = PY$.

Thus for any point P on the boundary of the yellow cross-section we have:

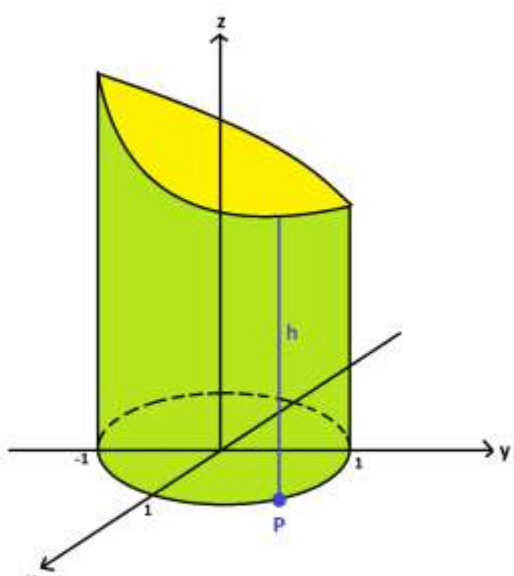
$$PF + PG = PX + PY = XY,$$

a fixed value. Thus this boundary curve does indeed satisfy the required distance property and is, indeed, an ellipse.

Comment: One can use this very same proof to establish that an ellipse, defined as a slice of a cone, does indeed satisfy this fixed distance sum property too:



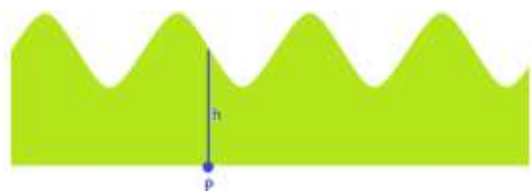
Question 2:
Let's set up a three-dimensional coordinate system with the rim of the cylinder the unit circle in the xy-plane and the length of the cylinder extending in the positive z-direction:



Suppose the top of the cylinder is sliced by some plane to reveal a yellow planar region as shown.

We are wondering the following:

For each point P on the unit circle in the xy-plane there is an associated height h from P to the boundary of the yellow cross-section. As P moves around the circle, does the value of h vary as a sine curve?



Suppose P has coordinates (x, y) in the xy-plane. We have: $x^2 + y^2 = 1$.

We need the equation of the slicing plane.

Now, in two-dimensional coordinate geometry the equation of a (non-vertical) line is given by $y = mx + b$ for some real numbers m and b. In three-dimensional coordinate geometry the equation of a (non-vertical) plane is given by $z = mx + ny + b$ for some real numbers m, n, and b. Let's suppose our slicing plane is given by such a formula.

So if P on the unit circle has coordinates (x, y) satisfying $x^2 + y^2 = 1$, then the height h we seek is simply:

$$h = z = mx + ny + b.$$

Is there trigonometry in this?

Yes!

It is better to present the coordinates of P on the unit circle in the xy-plane in terms of its argument θ from the positive x-axis. That is, let's write the coordinates of P as:

$$P = (\cos\theta, \sin\theta).$$

As θ grows, the point P does indeed traverse the unit circle.

Then the height we seek is given as:

$$h(\theta) = m\cos\theta + n\sin\theta + b\ .$$

Is this sure to be a simple sine curve? It doesn't look like one!

But if you read the December 2014 CURRICULUM ESSAY at www.jamestanton.com/wp-content/uploads/2012/03/Curriculum-Essay_December2014_TRIG-ADDITION.pdf then you know we are done: this is a simple sine curve in disguise!

To see this, use the trigonometric addition formula we derived in that essay.

$$A\sin(\theta + c) = A\sin\theta\cos c + A\cos\theta\sin c$$

The challenge is to choose values for A and c so that

$$A\sin(\theta + c) = m\cos\theta + n\sin\theta\ .$$

This suggests we want:

$$m = A\cos c$$

$$n = A\sin c$$

giving:

$$m^2 + n^2 = A^2$$

$$\tan c = \frac{n}{m}$$

And one can check that indeed $A\sin(\theta + c)$ does equal $m\cos\theta + n\sin\theta$ for $A = \sqrt{m^2 + n^2}$ and $c = \arctan\left(\frac{n}{m}\right)$.

So our height function:

$$h(\theta) = m\cos\theta + n\sin\theta + b$$
$$= A\sin(\theta + c) + b$$

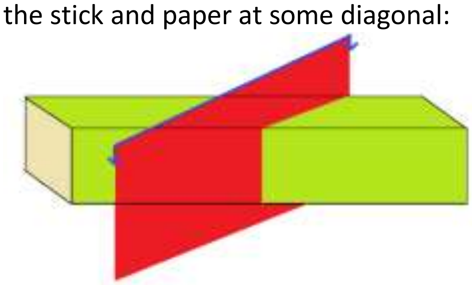
is indeed a beautiful sine curve.

So sometimes curves we see in nature do turn out to be what we might first naively guess!

RESEARCH CORNER

A stick of butter is a "square cylinder."

Suppose we wrap paper around and around a stick of butter, akin to wrapping-paper wrapped around a circular cylinder, and then cut through the stick and paper at some diagonal:



Question 1: *Is the cross-section sure to be in the shape of a parallelogram?*

If so, a parallelogram is then the "square version" of an ellipse. Where or what then are the two foci of a parallelogram?

Question 2: *If we unroll the paper from around the stick will we see the squine curve – the square version of sine – or some variation of it?*
(See www.jamestanton.com/?p=605 to learn about squine and cosquine.)

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