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Uplifting Mathematics for All

WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

DECEMBER 2020

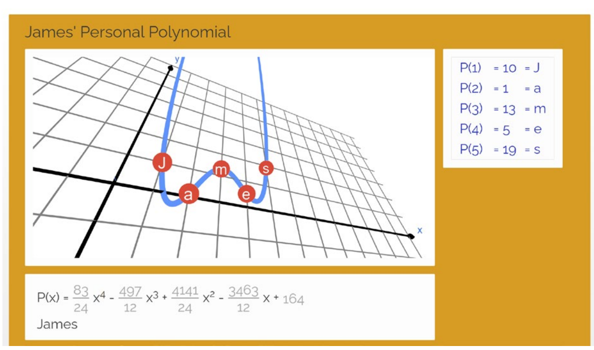
THIS MONTHS' PUZZLER:

1. A quadratic function has integer outputs for each of three consecutive integers as inputs. Prove that, actually, the quadratic function has integer outputs for *all* integer inputs.

2. A quadratic function has integer outputs for each of three distinct integer inputs (not necessarily consecutive). Prove the quadratic function has integer outputs for infinitely many integer inputs.

PERSONAL POLYNOMIALS

It's mighty fun to find a polynomial that spells your name. (Try it here: *Personal Polynomial*!)



In fact, it's possible to write down a polynomial formula that fits any desired set of data. For instance, consider this generic set of data.

x	a	b	c	d
y	A	B	C	D

Here's a polynomial that gives, for each input, the desired matching output.

$$P(x) = A\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} + B\frac{(x-a)(x-c)(x-d)}{(b-a)(b-c)(b-d)} + C\frac{(x-a)(x-b)(x-d)}{(c-a)(c-b)(c-d)} + D\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}$$

This looks scary, but when you start playing with the formula you see its construction is quite natural.

Check: *Put* $x = a$ *into the formula and see that all but one of the summands vanishes. And for the summand that survives, see that the denominator given was designed to cancel the numerator precisely when we* $x = a$. *This then gives*

$$P(a) = A\cdot 1 + 0 + 0 + 0 = A.$$

Examine too what happens for $x = b$, $x = c$, *and* $x = d$.

Practice: *Write down a quadratic function that fits the following data. (Don't bother simplifying your expression.)*

x	2	10	13
y	5	-8	4

One can use this technique (it's called *Lagrange Interpolation*) to write down a polynomial that fits any set of data. If there are k data points, a polynomial of degree $k-1$ results.

Practice: *Use this technique to write down the equation of the line that passes through the points* $(2,5)$ *and* $(7,12)$.

[It's $y = 5\cdot\frac{x-7}{(-5)} + 12\cdot\frac{x-2}{(5)}$.]

If each of the given inputs and outputs are integers, then the polynomial that results, when expanded out, has rational numbers as coefficients.

We have

Given a set of integer points in the plane, there is a polynomial function, with rational coefficients, whose graph passes through them.*

* Well, there is on caveat to this: to have a function—polynomial or otherwise—no two of these integer points may have the same x coordinate.

Practice: *What goes wrong if you try to apply Lagrange's Interpolation Formula to this data set?*

x	2	10	10
y	5	-8	4

Challenge:
"Two points determine a line."
This seems reasonable to believe.

In general, given k data values, prove that there is only one polynomial function of degree $k-1$ that fits that data. (Consequently, that polynomial function must be Lagrange's Interpolation polynomial.)

CONSECUTIVE INTEGER INPUTS

Suppose our given data has inputs that are consecutive integers.

x	a+1	a+2	a+3	a+4
y	A	B	C	D

Then matching Lagrange polynomial has some particularly nice structure.

$$\begin{aligned} P(x) = A \cdot & \frac{(x-a-2)(x-a-3)(x-a-4)}{(-1)(-2)(-3)} \\ + B \cdot & \frac{(x-a-1)}{(1)} \cdot \frac{(x-a-3)(x-a-4)}{(-1)(-2)} \\ + C \cdot & \frac{(x-a-1)(x-a-2)}{(2)(1)} \cdot \frac{(x-a-4)}{(-1)} \\ + D \cdot & \frac{(x-a-1)(x-a-2)(x-a-3)}{(3)(2)(1)} \end{aligned}$$

We see terms of the form

$$\pm \frac{(x-m-1)(x-m-2)\cdot \cdots \cdot (x-m-k)}{k!}.$$

If an input x is an integer, then the each of these terms is a product of k consecutive integers divided by $k!$, for some integer k.

Result: *The product of any k consecutive integers divided by $k!$ is an integer.*

Reason: For positive integers, observe that

$$\frac{(n+1)(n+2)\cdot \cdots \cdot (n+k)}{k!} = \frac{(n+k)!}{n!k!}$$

is the count of ways to choose k objects from a set of $n+k$ objects. This count is an integer!

If the set of consecutive integers includes 0, the result is true. And if all the integers are negative, then follow the same reasoning as above: we may just be off by a minus sign.

This establishes:

The Lagrange Polynomial for a set of data with consecutive integer inputs has integer values on all integer inputs.

In particular, we have explained the first puzzler.

SOME INTEGER INPUTS

Algebra shows

$$x^2 - a^2 = (x-a)(x+a)$$
$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$
$$x^4 - a^4 = (x-a)(x^3 + ax^2 + a^2x + a^3)$$

and, in general, that

$$x^k - a^k = (x-a)(polynomial)$$

for any integer $k > 0$.

If P is a polynomial with integer coefficients and a is an integer, it follows that

$$P(x) - P(a) = (x-a)(polynomial)$$

with the polynomial factor having integer coefficients as well.

Now suppose a polynomial of degree $k-1$ has integer outputs on k different integer inputs. This polynomial must be the Lagrange Interpolation polynomial for those k input/output pairs and so be a polynomial P with rational coefficients. By putting all these rational numbers over a common denominator, we can assume P has the form

$$P(x) = \frac{b_{k-1}x^{k-1} + \cdots + b_1 x + b_0}{d}$$

with $b_{k-1}, \ldots, b_1,\ b_0$ and d integers. We have that $Q(x) = b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$, a polynomial with integer coefficients, has integer value a multiple of d for at least k different integer inputs.

Let's prove that it is actually a multiple of d infinitely often.

Suppose $Q(a)$ is a multiple of d.
Consider $Q(a+d)$.

We have that $Q(a+d) - Q(a)$ is a multiple of $(a+d-a) = d$. It follows that $Q(a+d)$ is also a multiple of d.

By looking at $Q(a+2d) - Q(a+d)$ we see that $Q(a+2d)$ is a multiple of d too. And so are $Q(a+3d)$, $Q(a+4d)$, and so on.

Since Q adopts an integer value that is a multiple of d infinitely often, P takes on integer values infinitely often.

This establishes (and generalizes) the second puzzler.

RESEARCH CORNER

"Three points in space determine a plane."

Given k points in three-dimensional space, is there a polynomial function

$$P(x,y) = ax^k + bx^{k-1}y + \cdots + cxy^{k-1} + dy^k + \cdots + ex + fy + g$$

whose graph passes through each of them? Is there a "lowest-degree" polynomial that exists and is unique?

If a certain count of these points have integer coordinates, must the polynomial graph then pass through infinitely many points with integer coordinates?

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