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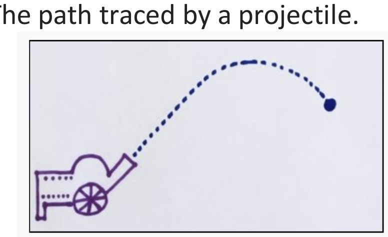
WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

DECEMBER 2019

THIS MONTHS' PUZZLER:
Which of the following curves are quadratic curves, that is, curves that when set on a coordinate plane can be described by the graph of an equation of the form $y = ax^2 + bx + c$ for some constants a, b, and c?

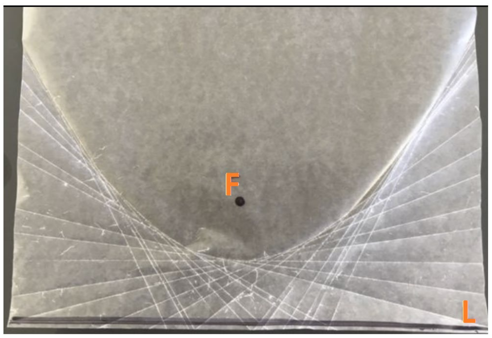
a) The path traced by a projectile.



b) The shape of a hanging chain.

c) The shape of a hanging chain being used to support a linear stretch of road beneath it, as for a classic suspension bridge.

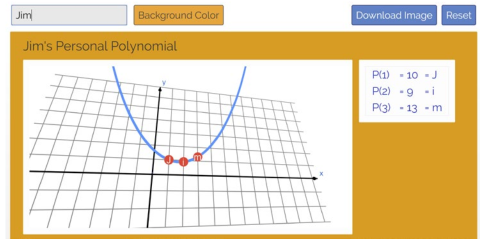
d) The classic Greek *parabola* defined as the locus of points equidistant from a given point F (the *focus)* and a given line L (the *directrix*) in the plane.



A Fun Activity: Draw a point F on a piece of wax paper and regard the bottom edge of the paper as the directrix line L. Bring the bottom edge up so that one point of that edge lies on F and make a crease. Repeat this action fifty times. What emerges, as the boundary of these crease marks, is a U-shaped curve. This curve is the set of points equidistant from F and L, that is, it is a Greek parabola. (See here for an explanation.)

e) The famous Arch of St Louis.

f) The formula that spells my nickname "JIM."



A Fun Activity: Do you know your *Personal Polynomial* ? If not, find it here! Type in JIM too if you are wondering what I mean by "the formula that spells my nickname."

ARE ALL U-SHAPED CURVES QUADRATIC?

In a typical school curriculum, one studies linear equations and their graphs for two or three years before—finally—taking a look at one example of an equation whose graph is not a straight line, namely, that of a quadratic equation. This is a U-shaped graph. As only this one example is typically discussed for the next year or two in an algebra curriculum, it is understandable then for students to naturally infer that all U-shaped curves must come from quadratic formulae. (After all, no other possibilities are presented.)

But which of the U-shaped curves in the opening puzzler are given by quadratic formulae? Only three of the six are! (And that count may be less if you don't want to make some simplifying assumptions!)

Projectile Motion

The motion of objects moving through the air solely under the influence of gravity is often given as the key motivator for studying quadratic equations.

It is understood that air resistance complicates matters. So, we tend to ignore the effect of having an Earthly atmosphere and imagine throwing things on the surface of the Moon instead perhaps.

Galileo (1564-1642) realized that the acceleration due to gravity falling objects experience is independent of their mass and is given by a constant value. (It's about 9.8 meters per squared second.) This led Galileo to then ascertain that propelled objects follow arcs of motion given by quadratic formulae. (See here on how to deduce this without calculus.)

So, it seems that the answer to part a) of the opening puzzler, assuming the absence of air resistance, is … YES: the arcs of projectile motion are given by quadratic formulae.

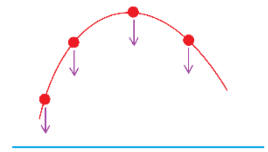
But even without air resistance this is not correct!

Objects propelled from the surface of an (atmosphere-less) planet follow elliptical arcs!

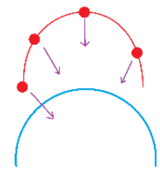
Galileo assumed in his work that force due to gravity operates in a fixed downward direction, as though the Earth is flat. But the Earth is essentially spherical and the direction of the gravity-force vector acting on a propelled object changes direction as the object moves: it changes to always points towards the center of the Earth.

Isaac Newton (1643-1727) proved, using calculus and his gravitational law, that the motion of heavenly bodies trapped in periodic orbits are elliptical. Planets revolving around the massive Sun experience a gravitational force that is always directed toward the center of the Sun. But the dynamics of a thrown object are the same: these objects are moving under the influence of the force of gravity directed to the center of the massive Earth. As such, projectile motion is the same as planetary motion and so is elliptical.

Surpsise!



Flat Earth.
Projectiles follow parabolic arcs.



Spherical Earth.
Projectiles follow elliptical arcs.

Hanging Chains and Suspension Bridges

Galileo wondered if the shape of a hanging chain is quadratic. His wonderings were swiftly proved incorrect by German scholar Joachim Junguis (1587-1657). But Jugnuis did not know the exact shape of the curve.

Fun Activity: Without knowledge of calculus, your students can too verify that a hanging chain is not following the shape of a quadratic graph. See lesson 3.4 here.

In 1691, in a response to a challenge posed by Jakob Bernoulli, mathematicians Gottfried Leibniz, Christiaan Huygens, and Johann Bernoulli each determined the exact shape of the curve. They proved it is given by an equation essentially of the form

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

This curve is today known as the *catenary* curve from the Latin word for chain.

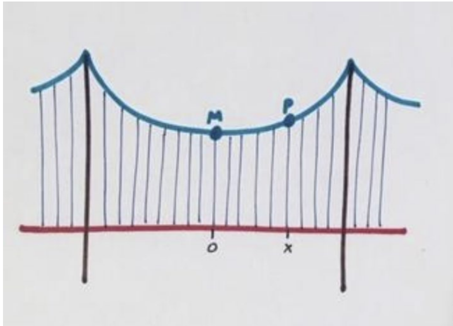
But here's the kicker:

Use a hanging chain to support a linear mass below it (as for a suspension bridge) and then the chain adjusts to adopts the shape of a quadratic graph!

Here's why.

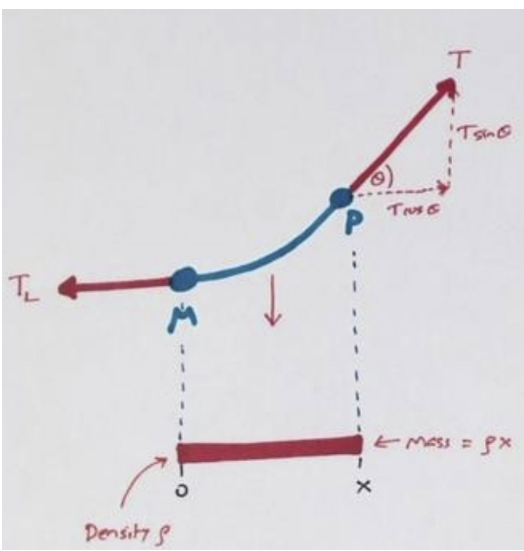
Warning: *A tiny piece of calculus is needed.*

A suspension bridge supports a heavy linear section of road beneath it: there are closely-spaced vertical cables that connect the road section to the main suspension cable – the "hanging chain."



We'll assume the weights of vertical cables and the main suspension cable are negligible compared to the weight of the road. We'll also assume that the spacing of the vertical cables is quite dense along the length of the road in the sense that every point on the suspension cable is very close to a vertical cable.

Consider a section of the suspension cable, from its lowest point M to some point P to its right. Let's say P is x meters to the right of this lowest point M .



This section of cable is experiencing—as a whole—three forces each trying to move it.

1. At M , there is the tension created by the left half of the suspension cable pulling the section of cable to the left. Let's label this force T_L for "leftward tension."

2. At P there is tension created by the right portion of the suspension cable pulling the our section upwards and to the right at some angle θ from the horizontal. Let's label this force T , simply for "tension."

3. The whole section is being pulled downward by the weight of the segment of road beneath it. If we assume the road is of uniform density ρ , say, and we have x meters of road, then the mass of the road is ρx and the downward force due to gravity is $\rho x g$, where g is the acceleration due to gravity.

Since our section of cable is not moving, these three forces balance.

The horizontal component of the second force T is $T\cos\theta$ and its vertical component is $T\sin\theta$.

Consequently, we have

$$T_L = T\cos\theta,$$
$$\rho x g = T\sin\theta.$$

It follows that

$$\tan\theta = \frac{T\sin\theta}{T\cos\theta} = \frac{\rho g}{T_L}x .$$

That is, $\tan\theta = kx$ for some fixed value k for all points on the right of the suspension cable.

But $\tan\theta$ is "rise over run" and so is the slope of the cable at the point P .

If the shape of the suspension cable is given by an equation of the form $y = f\left(x\right)$ for some formula of x s, we have just argued that

$$\frac{dy}{dx} = kx$$

for all points $P=(x,y)$ on the right half of the suspension cable.

Integrating gives $y=\frac{k}{2}x^2+c$ and so we have a quadratic shaped curve on the right, and by symmetry, on the left too!

Surprise! Suspension bridges follow quadratic shapes!

Comment: Using "difference methods" as outlined in the third course experience here we can side-step calculus and argue that $\tan\theta$ represents the changes in y-values over fixed differences in x-values, and so the y-values of the main suspension cable have differences given by a linear formula. Consequently, their double differences are constant and so the y-values are given be a quadratic formula.

Arches

In 1671, British scholar Robert Hooke announced he had determined the "optimal" shape of a free-standing arch, namely, that it matches the shape of a hanging chain, but inverted. I, personally, don't understand what "optimal" means here (a next essay?), but apparently inverted catenary shapes are commonplace for the structure of free-standing arches. At the very least, the famous St. Louis "gateway to the west" follows this catenary shape and is not a quadratic curve.

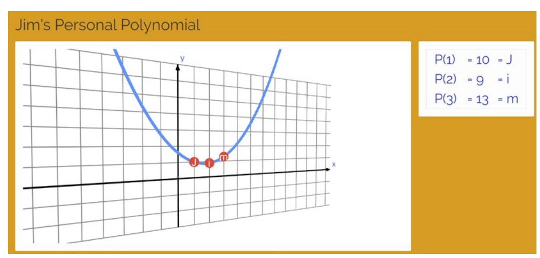
Personal Polynomials

Any three non-collinear points in the plane have a lie on a quadratic curve. For instance, here's the equation of quadratic equation that includes the data points (a,A), (b,B), and (c,C):

$$y=A\frac{(x-b)(x-c)}{(a-b)(a-c)}+B\frac{(x-a)(x-c)}{(b-a)(b-c)}+C\frac{(x-a)(x-b)}{(c-a)(c-b)}$$

If we expand each term and add like quantities, we see that this is indeed a quadratic equation. And one can readily check that inserting $x=a$ gives $y=A$, and so on. (We as assuming no two values a, b, c match.)

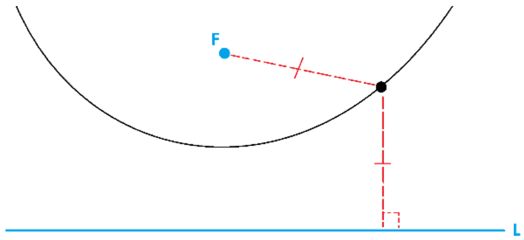
The data set $(1,10)$, $(2,9)$, $(3,13)$ gives a quadratic that "spells" JIM: Put in $x=1$, out comes 10 for the tenth letter of the alphabet (J); put in $x=2$, out comes 9 for the ninth letter of the alphabet (I); and put in $x=3$ and out comes 13 for the thirteenth letter (M). The Personal Polynomial for my nickname—and for any three-letter name—is quadratic.



A Fun Activity: This video shows how to write down a formula that fits any given set of data values in a table. Also look at the videos at the Personal Polynomial site.

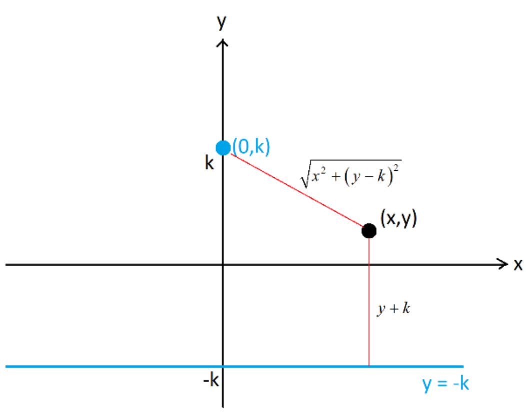
Parabolas

The Greek parabola is defined as the set of points equidistant from a given point F and line L in the plane.



For convenience let's assume L is a horizonal line k units below the x-axis in a coordinate plane and F is on the y-axis, k units above the x-axis. That is, L has equation $y=-k$ and F has coordinates $(0,k)$.

For a point $P=(x,y)$ to be equidistant from F and L we need $\sqrt{x^2+(y-k)^2}$ to equal $y+k$.



That is, we need x and y to satisfy

$$x^2+(y-k)^2=(y+k)^2$$

or, equivalently,

$$y=\frac{1}{4k}x^2,$$

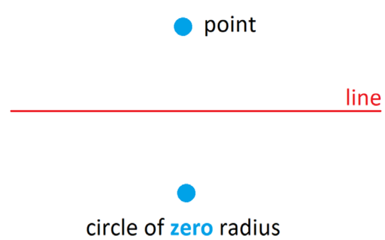
a quadratic formula. And, conversely, any point with coordinates (x,y) satisfying $y=\frac{1}{4k}x^2$ has

$$\sqrt{x^2+(y-k)^2}=y+k$$

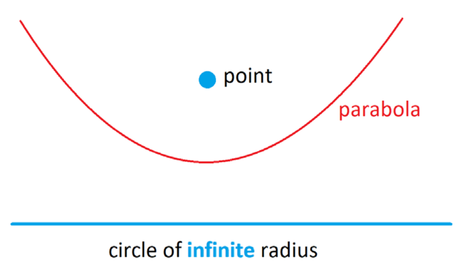
and so is a point equidistant from F and L .

CHALLENGE 1: Is the graph of every quadratic equation $y=ax^2+bx+c$ a parabola? How do you know? If the answer is yes, can you give the coordinates of its focus F and the equation of its directrix L?

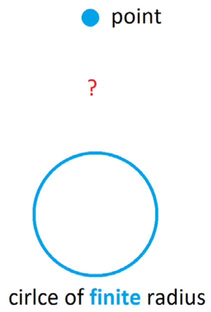
CHALLENGE 2: One learns in geometry class that the locus of points equidistant from two given points is a line (the perpendicular bisector of line segment connecting those points). Regard the second point as a circle of zero radius.



We've just seen that the locus of points from a point and a line is a parabola. Regard the line as a circle of infinite radius.



What about the intermediate case? What is the locus of points equidistant from a point and a circle of finite radius? (How does one measure the distance of a point from a circle?)



Go Further: What is the locus of points equidistant from two circles of different radii? What if the circles intersect or if one sits inside the other?

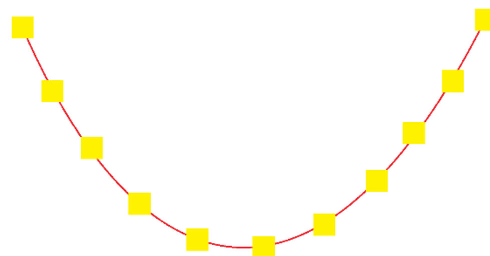
Answer Shocker: The locus of points is a parabola each and every time!

RESEARCH CORNER

In our analysis of suspension bridges, we assumed that there is a vertical cable supporting the underlying road section at each and every location along the road.

Let's not assume this.

Suppose we place heavy weights at evenly space positions along a chain of negligible mass. On what curve do the center of masses of each of the weights lie?



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