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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

AUGUST 2020

Here's a reprise of a favorite topic. I wrote about this accomplishment of three high school students in MATHEMATICS GALORE (MAA, 2012), but further ideas and questions about their approach have been swirling in my mind this month.

THIS MONTHS' PUZZLER:
Consider your favorite sequence of positive integers, say, the Fibonacci numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

One caveat: Choose a sequence that does not decrease in value from any one term to the next. Values may repeat but, overall, your sequence should increase.

Now ask:

How many terms in the sequence have value less than 1? **0**
How many terms in the sequence have value less than 2? **2**
How many terms in the sequence have value less than 3? **3**
How many terms in the sequence have value less than 4? **4**

How many terms in the sequence have value less than 5? **4**
How many terms in the sequence have value less than 6? **5**
How many terms in the sequence have value less than 7? **5**
How many terms in the sequence have value less than 8? **5**
How many terms in the sequence have value less than 9? **6**
And so on.

This gives the frequency sequence of the sequence.

0, 2, 3, 4, 4, 6, 6, 6, 6, 6, 6, 6, 6, 7, …

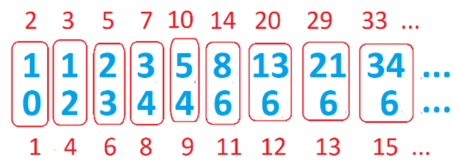
Now do it again! Take the frequency of the frequency sequence.

How many terms in the sequence have value less than 1? **1**
How many terms in the sequence have value less than 2? **1**
How many terms in the sequence have value less than 3? **2**
How many terms in the sequence have value less than 4? **3**
How many terms in the sequence have value less than 5? **5**
How many terms in the sequence have value less than 6? **8**
How many terms in the sequence have value less than 7? **13**
And so on.

WHOA! We're back to the original Fibonacci sequence!

But wait! More is true.

Add to each first term of the sequence and the frequency sequence the position number 1; to each second term the position number 2; to each third term the position number 3; and so on.



It seems that each and every counting number, 1, 2, 3, 4, 5, 6, 7, … appears exactly once - no repetition.

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There is nothing special about the Fibonacci numbers here.

Take any non-decreasing sequence of counting numbers.

Then the frequency sequence of its frequency sequence is the original sequence.

$\{S_n\}$: 1 2 2 2 3 3 6 7 7 7 9 11 11 14 15 ...
$\{FS_n\}$: 0 1 4 6 6 6 7 10 10 11 11 13 13 13 14 ...
$\{FFS_n\}$: 1 2 2 2 3 3 6 7 7 7 9 11 11 14 15 ...

Moreover, add the position number to each term of the sequence and its frequency sequence and all the counting numbers 1, 2, 3, 4, … appear without repetition.

$\{S_n + n\}$: 2 4 5 6 8 9 13 15 16 17 20 23 24 28 30...
$\{FS_n + n\}$: 1 3 7 10 11 12 14 18 19 21 22 25 26 27 29 ...

What's going on?

EXPLANATION

This curious phenomenon about non-decreasing sequences was discovered back in 1954 by J. Lambeck and L. Moser. But in 2005, three then high-school students, E. Rudyak, J.S. You, and C. Zodda, developed a purely visual approach explaining the result.

Here's their approach.

Any non-decreasing sequence $\{S_n\}$ of counting numbers can be encoded as a string of dots and dashes. For example, the sequence

1 2 2 2 3 3 6 7 7 7 9 11 11 14 15 …

can be encoded as shown.

Here, the nth term of the sequence is:

$S_n =$ the number of dots to the left of the nth dash.

<u>Check</u>: The 10th dash in the picture has seven dots to its left and indeed, $S_{10} = 7$. And can you see $S_{11} = 9$?

Practice: *Draw a dots and dashes representation of the Fibonacci sequence.*

Now comes the mind-twisty part!

FS_n is the number of terms in the original sequence with value smaller than n.

This is the number of dashes with less than n dots to their left.

Each dash to the left of the nth dot has less than n dots to its left. So

FS_n is the number of dashes to the left of the nth dot.

Summary:

$S_n =$ number of **dots** to the left of the nth **dash**.

$FS_n =$ number of **dashes** to the left of the nth **dot**.

Taking the frequency just interchanges the roles of dots and dashes in the definition of the picture.

So, the frequency of the frequency sequence interchanges the words "dot" and "dash" again and returns us to the original picture definition.

$FFS_n =$ number of **dots** to the left of the nth **dash**,

which is S_n.

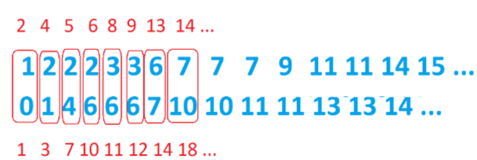
Going further, look at the position numbers of each dot and dash.

Below the dashes we have the numbers:

$2\ 4\ 5\ 6\ 8\ 9\ 13\ 15\ 16\ 17\ 20\ 23\ \ 24\ \ 28\ 30 \ldots$

Below the dots we have the numbers:
$1\ 3\ 7\ 10\ 11\ 12\ 14\ 18\ 19\ 21\ 22\ 25\ 26\ 27\ \ 28 \ldots$

And compare this to adding the position number of each term of the original sequence and its frequency sequence.



$\{S_n\}$: 1 2 2 2 3 3 6 7 7 7 9 11 11 14 15 ...
$\{FS_n\}$: 0 1 4 6 6 6 7 10 10 11 11 13 13 13 14 ...

$\{S_n + n\}$: 2 4 5 6 8 9 13 15 16 17 20 23 24 28 30...
$\{FS_n + n\}$: 1 3 7 10 11 12 14 18 19 21 22 25 26 27 29 ...

What's going on now?

A key unlocking this is to ask something like: *What is the position number of the 10th dash?*

Well …

S_{10} is, by definition, the number of dots to the left of the 10th dash.

So, the 10th dash has S_{10} dots to its left.

Also, there are 9 dashes to the left of the 10th dash.

So, there are $S_{10} + 9$ objects to the left of the 10th dash.

Thus the 10th dash is in position $S_{10} + 9 + 1 = S_{10} + 10$.

Check: *What is the value of S_{10} for our sequence? Is the tenth dash in the purple picture indeed in the $(S_{10} + 10)$ th position?*

In general:

The n th dash is in position $S_n + n$.

Since the picture of the frequency sequence is the same picture but with the roles of dots and dashes interchanged, we must have

The n th dot is in position $FS_n + n$.

So as we run through all the position numbers 1, 2, 3, 4,5, 6, 7, … , those numbers matching dashes make the sequence $\{S_n + n\}$ and those matching dots make the sequence $\{FS_n + n\}$.

Wild!

APPLICATION

Here are the square numbers:
1, 4, 9, 16, 25, 36, …
Here are the non-square numbers:
2, 3, 5, 6, 7, 8, 10, 11, …

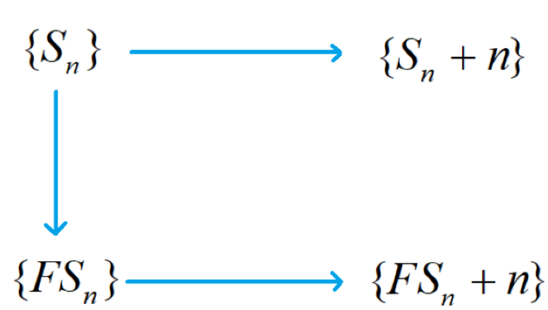
There's a formula for the n th square number, namely, n^2.

Is there a formula for the n th non-square number?

Why yes!

The process we've just described gives a means to split the counting numbers 1, 2, 3, 4, … into two complementary sequences.

Start with any non-decreasing sequence $\{S_n\}$. Find its frequency sequence $\{FS_n\}$. Adding position numbers to the terms of each gives complementary sequences $\{S_n + n\}$ and $\{FS_n + n\}$.



Choose $\{S_n + n\}$ to be the sequence 1, 4, 9, 16, 25, 36, …..

Then $\{FS_n + n\}$ will be the sequence 2, 3, 5, 6, 7, 8, 10, 11, ….

We have a formula for the terms of $\{S_n + n\}$. The nth term is $S_n + n = n^2$. So

$$S_n = n^2 - n.$$

Now FS_n is the number of these values that are less than n. That is,

$FS_n =$ number of k such that $k^2 - k < n$.

Here's something sneaky: Since we're dealing with integers, $k^2 - k < n$ only if $k^2 - k + \frac{1}{4} < n$. So

$FS_n =$ number of k such that

$$k^2 - k + \frac{1}{4} < n.$$

That is,

$FS_n =$ number of k such that

$$\left(k - \frac{1}{2}\right)^2 < n$$

and so

$FS_n =$ number of k such that $k < \sqrt{n} + \frac{1}{2}$.

That is, FS_n is the value of the largest integer just below or equal to $\sqrt{n} + \frac{1}{2}$.

Exercise: *Show that the largest integer less than or equal to $x + \frac{1}{2}$ is $< x >$, the number x rounded up or down to the closest integer.*

We have that

$$FS_n = < \sqrt{n} >$$

and so the nth non-square number is given by the formula

$$FS_n = n + < \sqrt{n} >.$$

Whoa!

Challenge: The sequence of triangular numbers begins 1, 3, 6, 10, 15, 21, 28, …. The nth triangular number has formula $\frac{n^2+n}{2}$. Show that the nth non-triangular number is given by $n + < \sqrt{2n} >$.

Challenge: The sequence of oblong numbers begins 2, 6, 12, 20, 30, 42, 56, …. The nth oblong number has formula $n(n+1)$. Show that the nth non-oblong number is given by $n + \sqrt{n}$ rounded down to the nearest integer strictly less than $n + \sqrt{n}$.

RESEARCH CORNER

1. The nth non-square number is given by $n + < \sqrt{n} >$. The nth non-triangular number is given by $n + < \sqrt{2n} >$.

What non-numbers are given by $n + < \sqrt{3n} >$? Can you make sense of these numbers?

What non-numbers are given by $n + < \sqrt{1.5n} >$, halfway between being non-square and non-triangular? Can you make any sense of these too?

2. Find formulas for other non-number sequences if you can.

What's the n th non-Fibonacci number?
What's the n th non-cube number?

Warning: This is much harder than one would hope. If you have any success in finding a formula for any interesting non-number sequence, let me know!

3. Suppose you are allowed to repeat a value infinitely often.

Consider, say,

$\{S_n\}: 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ ...$

Then

$\{FS_n\}:\ 0,\ 1,\ w,\ w,\ w,\ ...$

where w is Cantor's first transfinite ordinal. Do we still have a working theory?

How about sequences like this?

$\{S_n\}: 1\ 2\ 2\ 2\ 2\ 2\ ...\ 3\ \ 6\ 6\ 9\ 9\ 9\ 9\ 9\ 9\ ...\ 13\ 16\ 17\ 17\ 17\ \ 22\ \ 23\ \ 23\ 23\ 23....$

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