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WHAT COOL MATH!

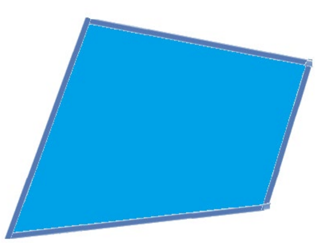
CURIOUS MATHEMATICS FOR FUN AND JOY

AUGUST 2019

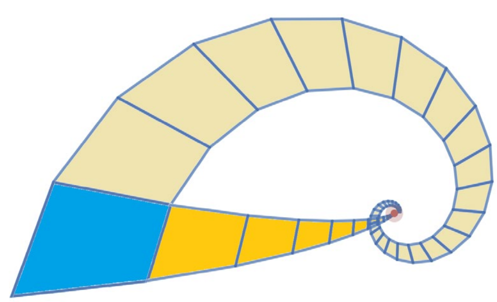
TODAY'S PUZZLER

Last month I presented this puzzler from David Henderson.

Start with a single quadrilateral with no two opposite sides equal in length.



On the shorter side of each opposite pair, construct an "arm" of scaled copies of this quadrilateral, no flipping of orientation.



Prove that these two arms are sure to converge to a common point in the plane.

In that essay I shared a proof using complex number theory, but at the end of the piece I called for a simpler, swifter, and more elegant proof.

For this month's puzzler…
Find a simpler, swifter, and more elegant proof—one that does not rely complex numbers!

GETTING THE GEOMETRIC GEARS TURNING

Within days of sharing last month's essays, two people wrote to me with purely geometric proofs of Henderson's result. Dr. Sam V. explained to me the concept of "spiral symmetry"—two figures are spirally symmetric if one can be mapped to the other via a rotation and dilation about a given point. He demonstrated that all the quadrilaterals in any one arm are spirally symmetric about a common point and, moreover, converge to that point. He then went on to prove that centers of spiral symmetry for each arm must be the same point.

Nathaniel A., a mathematics major at a university in Singapore, also taught me the concept of spiral symmetry and, moreover, of the "4-Miquel point." He proved that if the four sides of the given quadrilateral are extended to lines, then each of the two arms must converge to the Miquel point of those four lines.

Both emails were exciting to receive. Moreover, each email got my brain into thinking of purely geometric approaches to matters. What do you think of this next proof of Henderson's charming result?

A TWO-STEP PROOF

We establish

> <u>STEP 1</u>: **Each arm of similar quadrilaterals is sure to converge to a point in the plane.**

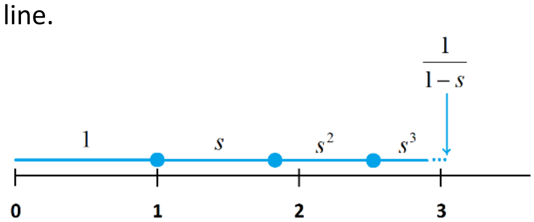
and

> <u>STEP 2</u>: **The two arms cannot converge to different points.**

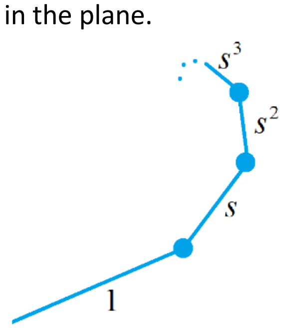
The first step was covered in last month's essay, but it need not have mentioned complex numbers at the time. After all, the geometric formula

$$1 + s + s^2 + s^2 + \cdots = \frac{1}{1-s}$$

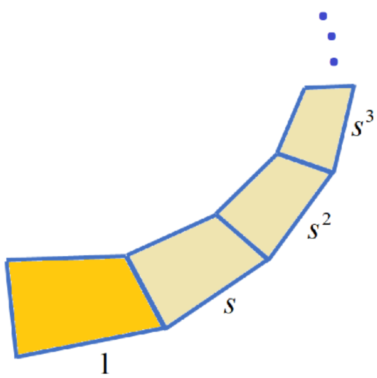
for a <u>real</u> number $0 < s < 1$ shows that, in a mind experiment, if we could lay down on the number line, end-to-end, an infinite collection of rods of lengths 1, s, s^2, s^3, … they would "reach" the point $\frac{1}{1-s}$ on the line.



Consequently, if we bend this string of rods at each connection point to make a spiral (or any other pattern) in two-dimensional space, the string of line segments converges to a point in the plane.

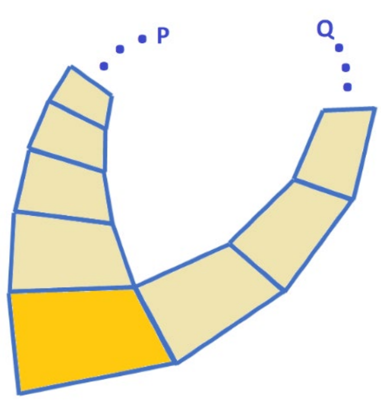


Now consider one arm of quadrilaterals from the puzzle. Let $0 < s < 1$ be the ratio of side lengths of the pair of opposite sides on which the arm is constructed. Then each quadrilateral in the arm is a scaled copy of its previous neighbor, scaled by s, and so the areas of these quadrilaterals, changing by a factor of s^2 each time, converge to zero. If we take the bottom edge of the original quadrilateral to be of unit length, then collection of bottom edges of the quadrilaterals along the arm have lengths 1, s, s^2, s^3, …. and so converge to a point in the plane. And since their areas of converge to zero, the entire arm converges to this same point. (My language is a bit loose here, but the idea is clear.)

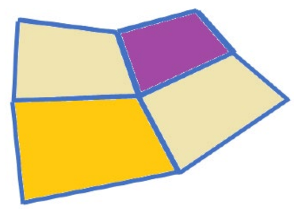


Now to step 2.

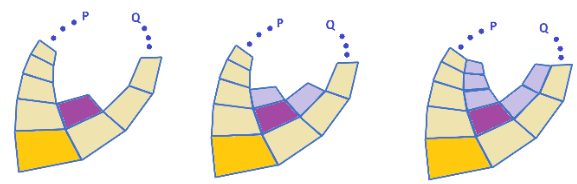
Suppose the arms of the quadrilateral converge to two points P and Q in the plane. Let d be distance between them. We will show that this distance must be zero.



To do this, notice that between any three scaled copies of the original quadrilateral arranged in an L-shape, we can insert a fourth scaled copy as shown. (Think through this.)



This means, between the two arms of our diagram we can insert another picture of two arms converging to the same two points P and Q.



This new set of arms is a scaled copy of a picture of the original two arms, scaled by some factor k. (The value k is the product of the two scale factors used for each of the original arms.) In which case, the distance between the two points P and Q must also be scaled by k. The equation $d = kd$ can only hold if $d = 0$. Voila!

RESEARCH CORNER

A triangle is a quadrilateral with one side of zero length. Are there any interesting patterns or results to be explored with "arms" constructed on triangles?

When might arms constructed on two sides of a pentagon converge to a common point?

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