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CURIOUS MATHEMATICS FOR FUN AND JOY

APRIL 2021

THIS MONTHS' PUZZLER:
How many numbers with at most two-digits possess a 1 as one of its digits? How many numbers of at most three-digits possess a 1 as one of its digits?

In the long run, what percentage of counting numbers possess at least one digit that is a 1?

In the long run, what percentage of counting numbers possess at least one of each digit 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9?

COUNTING 1s

There are $10^N - 1$ counting numbers with at most N digits.

A complicated way to see this is to allow leading zeros in our considerations, that is, to regard any number with less than N digits as an N-digit number with an initial set of zeros. So, to create an N-digit number (possibly with leading zeros) we need to fill each of N places with one of ten options—a digit from 0 through 9. This gives 10^N possibilities in all, except, we must exclude the one case of having all zeros.

The same reasoning shows that there are $9^N - 1$ numbers of most N digits with no digit equal to 1. (Fill each of N slots with one of nine choices: 0, 2, 3, 4, 5, 6, 7, 8, or 9.)

Consequently

$$\left(10^N - 1\right) - \left(9^N - 1\right) = 10^N - 9^N$$

numbers at most N digits long possess at least one digit equal to 1.

> **Challenge:** There is nothing too special about the digit 1 here.
> Among the $10^N - 1$ numbers at most N digits long, $10^N - 9^N$ of them have a digit of 2, and $10^N - 9^N$ of them have a digit of 7, and $10^N - 9^N$ of them have a digit of k for any given non-zero digit k.
>
> But how many numbers at most N digits long possess a (non-leading) digit of 0?

> **Challenge:** Among all the numbers at most N digits long, $8^N - 1$ of them fail to possess a digit of either 1 or 2; and $7^N - 1$ of them fail to possess a digit of either 1, 2, or 3.
>
> Show that $8^N - 7^N$ of them fail to possess a digit of 1 or 2, but do contain a digit of 3.
>
> Show that $9^N - 2\cdot 8^N + 7^N$ of them fail to possess of digit of 1, but do possess a digit of 2 and do possess a digit of 3.

The proportion of numbers at most N digits long that <u>do</u> possess a digit of 1 is

$$\frac{10^N - 9^N}{10^N - 1} = \frac{1 - (9/10)^N}{1 - (1/10)^N}.$$

This approaches the value $\frac{1-0}{1-0} = 1$ as N grows. So, in this sense, we can say:

In the long run, essentially all of the counting numbers possess at least one digit of 1.

> **Question:**
> Is the same true for the digit 0?

> **Question:** What proportion of numbers exactly N digits long possess at least on digit of 1? Does this proportion grow to 100% as N grows? (Is the same true for the digit 0?)

THE SET OF ONE-LESS NUMBERS IS SMALL

The set of all counting numbers is big, at least in the sense that the sum of their reciprocals fails to converge.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \text{ is infinitely large.}$$

To see this, observe, for instance

$$\begin{aligned}
&1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{8}\right) \\
&\qquad + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{8}\right) \\
&\qquad + \left(\frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots
\end{aligned}$$

The set of powers of two, $\{1, 2, 4, 8, 16, \ldots\}$, by contrast, is small in the sense that the sum of their reciprocals is not infinite.

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{1}{1 - 1/2} = 2 < \infty$$

Is the set of counting numbers that don't possess a digit of 1 small in this sense too? The answer is YES!

Result: The set of "one-less" counting numbers is small.

Proof: Consider the sum of the reciprocals of all the counting numbers that don't possess a digit of 1. Call the sum S. It begins

$$S = \left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{9}\right)+\left(\frac{1}{20}+\cdots+\frac{1}{99}\right) + \left(\frac{1}{200}+\cdots+\frac{1}{999}\right)+\cdots$$

There are 8 one-less one-digit numbers, and $8\cdot 9$ one-less two-digit numbers, and $8\cdot 9^2$ one-less three-digit numbers, and so on. Thus

$$\begin{aligned} S &< \left(\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}\right)+\left(\frac{1}{20}+\cdots+\frac{1}{20}\right) + \left(\frac{1}{200}+\cdots+\frac{1}{200}\right)+\cdots \\ &= 8\cdot\frac{1}{2}+8\cdot 9\cdot\frac{1}{20}+8\cdot 9^2\cdot\frac{1}{200}+\cdots \\ &= 4+4\left(\frac{9}{10}\right)+4\left(\frac{9}{10}\right)^2+\cdots \\ &= 4\cdot\frac{1}{1-\frac{9}{10}} \\ &= 40. \end{aligned}$$

The sum of one-less numbers is not large enough to have a sum of reciprocals that grows to be infinite!

Challenge: Show that the set of zero-less numbers is also small in this sense.

RESEARCH CORNER

Is the set of counting numbers missing any one or more of the digits 0, 1, 2, …, 8, or 9 "small" in the sense of the previous section?

A Theoretical Question

Let A be a set of counting numbers that is "small" in the sense of the previous section:

$$\sum_{n\in A}\frac{1}{n} < \infty .$$

Does this mean that "most" counting numbers are not in A in the sense of the first section of this essay, namely, that

$$\lim_{N\to\infty}\frac{\text{Number of numbers } \le N \text{ not in } A}{N}=0\,?$$

Does the converse hold?

(Actually … have you caught a logical slip here? The condition

$$\lim_{N\to\infty}\frac{\text{Number of numbers } \le N \text{ not in } A}{N}=0$$

with A the set of one-less numbers was not actually proved in the first section of this essay! We established this limit only for N running through the values 9, 99, 999, 9999, …. .

Can you complete the details and make the ideas in this essay robust and logically tight?)

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