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SUMMER 2021 BONUS

BONUS ISSUE

When recently looking into triangles posessing integer side lenghs and integer area—Heronian triangles—I was led down a rabbit hole of trying to understand various claims stated on websites and in the standard literature about them. Claims, it seems, are often made without proof or just with cryptic hints to (possible) proofs.

This essay is my attempt to put together in one spot a mathematically robust compendium of results.

HERON'S FORMULA

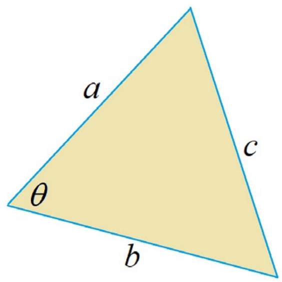
As mentioned in my June 2021 essay, the following formula for the area of a triangle expressed solely in terms of the side lengths of the triangle is attributed to Hero of Alexandria (ca. 10 CE – 70 CE)

Heron's Formula: If a triangle has side lengths a, b, and c, then its area A is given by

$$A=\sqrt{\frac{a+b+c}{2}\cdot\frac{-a+b+c}{2}\cdot\frac{a-b+c}{2}\cdot\frac{a+b-c}{2}}$$

One way to obtain this formula is to make use of two results from trigonometry. If θ is the angle between the sides labeled a and b, then we have

$$A = \frac{1}{2}ab\sin(\theta)$$
$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$



The relation

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

then gives

$$\left(c^2 - a^2 - b^2\right)^2 + 16A^2 = 4a^2b^2 .$$

Thus $16A^2$ is a difference of two squares and we have

$$16A^2 =$$
$$\left(2ab - c^2 + a^2 + b^2\right)\left(2ab + c^2 - a^2 - b^2\right)$$

The first term in the product is

$$\left((a+b)^2 - c^2\right) =$$
$$(a+b-c)(a+b+c)$$

and the second term is

$$\left(c^2 - (a-b)^2\right) =$$
$$(-a+b+c)(a-b+c)$$

Thus

$$16A^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$

giving Heron's formula.

HERONIAN TRIANGLES

A triangle with integer side lengths and possessing integer area A is today called *Heronian*. Actually, people tend to broaden this definition a little.

Definition: A triangle with side lengths and area each a rational value is called Heronian.

Any Heronian triangle in this sense can be scaled to one with integer side lengths and integer area: simply scale the triangle by a common denominator of all four rational numbers a, b, c, and A.

We'll call Heronian triangle with integer side lengths and area integer Heronian.

As we saw in the June 2021 essay:

Result 1: Every integer right triangle is integer Heronian.

Result 2: There are infinitely many examples of integer Heronian triangles. (We showed that there are infinitely-many almost-equilateral integer Heronian triangles, in particular.)

Going further …

Result 3: The perimeter of any integer Heronian triangle is an even integer.

Reason: Suppose, to the contrary, that $a+b+c$ is odd. Then each of the integers

$$-a+b+c = a+b+c-2a$$
$$a-b+c = a+b+c-2b$$
$$a+b-c = a+b+c-2c$$

is also odd, and so, from Heron's formula, $16A^2$ is the product of four odd numbers, and so is odd. This is patently not the case!

Corollary 4: For any integer Heronian triangle with side lengths a, b, c the value $s=\dfrac{a+b+c}{2}$ is an integer.

The value s is called the semi-perimeter of the triangle. Heron's formula can be presented as

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

with the quantity under the square root a product of four integers for integer Heronian triangles.

Corollary 5: There is no integer Heronian triangle with a side of length 1. Nor is there one with a side of length 2.

Reason: Any two sides of a triangle must have lengths that sum to more than the length of the third side. Thus, if an integer triangle has two sides of lengths 1 and a, the third side must also have integer length a. But then $1+a+a$ is not even, and so the triangle cannot be Heronian.

If an integer triangle has two sides of length 2 and a, the third side has length either a or $a+1$.

In the second case, $2+a+a+1$ is not even and so the triangle cannot be Heronian. In the first case, the triangle has area A satisfying
$A^2=(a+1)(1)(1)(a-1)=a^2-1$.
But no two non-zero square numbers differ by 1, so this too is impossible.

Fun Fact 6: The semi-perimeter of any integer Heronian triangle is a composite number.

Reason: We have

$$A^2=s(s-a)(s-b)(s-c).$$

In the prime factorization of A^2, any prime that appears must do so an even number of times since A^2 is square. If s were prime, then s must consequently appear an even count of times in the right side of the expression above. But each of the values $s-a$, $s-b$, and $s-c$ is less than s and so this cannot be the case. It must be then that s is composite.

Result 7: For each interior angle θ of a Heronian triangle $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ are rational numbers.

Reason: Using the notation of the previous section, if a, b, c and A have rational values, then so too do $\sin(\theta)=\dfrac{2A}{ab}$,

$\cos(\theta)=\dfrac{c^2-a^2-b^2}{2ab}$ and

$\tan(\theta)=\dfrac{\sin(\theta)}{\cos(\theta)}$.

Corollary 8: No Heronian triangle possesses an angle of 30°, 45°, or 60°.

Result 9: Any triangle with the sine of each angle in the triangle a rational number is similar to a Heronian triangle. (If one side length of the triangle is rational, then the triangle is Heronian.)

Reason: Suppose a triangle has angles α, β, and γ opposite sides of lengths a, b, and c, respectively. Scale the triangle so that the length a is rational.

Then the law of sines gives

$b = a \cdot \frac{\sin(\alpha)}{\sin(\beta)}$ and $c = a \cdot \frac{\sin(\gamma)}{\sin(\alpha)}$

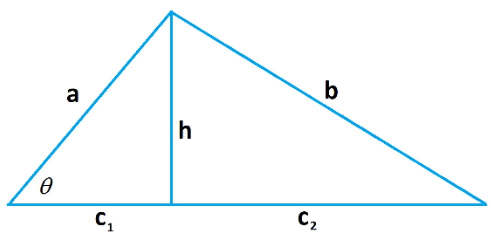
and the area of the triangle is

$A = \frac{1}{2} bc \sin(\gamma)$.

These are all rational values.

Result 10: Each altitude in a Heronian triangle has rational length. If the altitude meets a side of the triangle, it divides its length into two sections, each of rational length.

Reason: Suppose an altitude of a Heronian triangle with side lengths a, b, and c has length h and meets the side of length c, dividing it into sections of lengths c_1 and c_2, as shown.

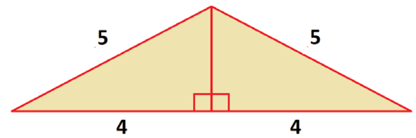


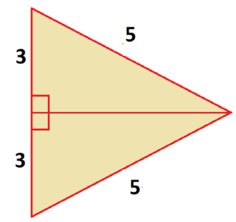
Let θ be the angle shown. From result 7, $\sin(\theta)$ and $\cos(\theta)$ are rational. Thus $h = a\sin(\theta)$ and $c_1 = a\cos(\theta)$ and $c_2 = c - c_1$ are rational too. (Also, $A = \frac{1}{2}ch$ shows that h must be rational.)

This reasoning applies if the triangle is oblique and the altitude of the triangle lies outside the triangle.

PASTING TOGETHER RIGHT TRIANGLES

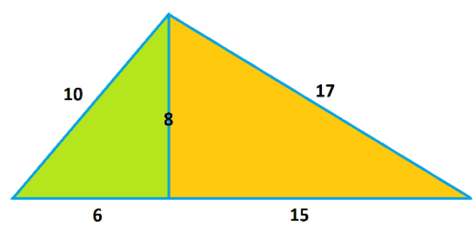
If two integer right triangles share a common leg length (not hypotenuse), then pasting together the two triangles along that common leg gives an integer Heronian triangle. For example, we create the 5-5-6 and 5-5-8 isosceles integer Heronian triangles by pasting together two copies of the 3-4-5 right triangle.





As there are infinitely many integer right triangles, there are infinitely many isosceles integer Heronian triangles.

Pasting together the 6-8-10 and 8-15-17 right triangles produces the 10-17-21 integer Heronian triangle.



Euclid (ca. 300BCE) established that every integer right triangle has side lengths given by

$$k(p^2 - q^2)$$
$$k(2pq)$$
$$k(p^2 + q^2)$$

for some integers k, p, q with p and q sharing no common factor. The area of such a triangle is the integer

$$\frac{1}{2} \cdot k(p^2 - q^2) \cdot k(2pq) = k^2 pq(p^2 - q^2).$$

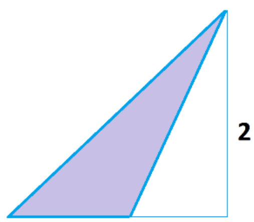
If we scale each side length down by the factor kpq, then we see that every integer right triangle is similar to one with rational side lengths

$$\frac{p^2 - q^2}{pq},\ 2,\ \frac{p^2 + q^2}{pq}$$

and rational area $\frac{p^2 - q^2}{pq}$.

Any two such rational right triangles can be pasted together along the sides of length 2 and the figure can be rescaled to then produce an integer Heronian triangle.

One can also “subtract” two such triangles to obtain oblique integer Heronian triangles.



Leonard Dickson in his volume *History of the Theory of Numbers, Vol II* (New York: Dover 2005) suggests that Euler (1707-1783) may have taken this approach to classify all Heronian triangles, and that before this, Brahmagupta (ca. 590 – ca. 668) may have conducted a similar approach. After all, result 10 shows that every Heronian triangle is composed to two right triangles with rational side lengths pasted together along a common edge.

Many standard websites, however, present a different classification of Heronian triangles without clear attribution ([Wikipedia]() and [MathWorld](), for instance). The stated classification appears in Robert Carmichael’s 1915 volume *Doiphante Analysis* (New York: Whiley and Sons, 1915), which is the earliest source of it I have been able to trace. Some websites imply that this exact version of the classification is Brahmagupta’s classification, but I have not been able to find clarity on this.

Classification of Heronian Triangles 11

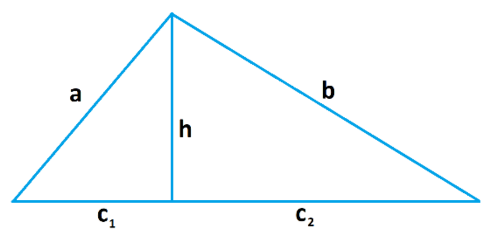
Every Heronian triangle is a scaled version of an integer Heronian triangle with sides

$$a = n(m^2 + h^2)$$
$$b = m(n^2 + h^2)$$
$$c = (m + n)(mn - h^2)$$

where m, n, and h are positive integers with $mn > h^2$.

This particular Heronian triangle has area
$$A = hmn(m+n)(mn - h^2).$$

Reason: Consider a Herorian triangle with rational side lengths a, b, c, rational area A, and draw an interior altitude of length h. Suppose this altitude meets the side of length c dividing it into sections of length c_1 and c_2 as shown.



We have that $h = \frac{2A}{c}$, c_1 and c_2 are rational by result 10.

Let m denote the rational number $a + c_1$, and n the rational number $b + c_2$.

We have that

$$h^2 = a^2 - c_1^{\ 2} = m(a - c_1)$$
$$h^2 = a^2 - c_2^{\ 2} = n(a - c_2)$$

Adding and subtracting the pair of equations

$$a + c_1 = m$$
$$a - c_1 = \frac{h^2}{m}$$

gives, respectively,

$$a = \frac{m + \frac{h^2}{m}}{2}, \quad c_1 = \frac{m - \frac{h^2}{m}}{2}.$$

Similarly

$$b = \frac{n + \frac{h^2}{n}}{2}, \quad c_2 = \frac{n - \frac{h^2}{n}}{2}$$

Also,

$$c = c_1 + c_2 = \frac{m - \frac{h^2}{m} + n - \frac{h^2}{n}}{2} = \frac{(m+n)(mn - h^2)}{2mn}.$$

Scale the triangle by the factor $2mn$ to see that our Heronian triangle is thus similar to one with sides

$$n(m^2 + h^2)$$
$$m(n^2 + h^2)$$
$$(m+n)(mn - h^2)$$

for three rational numbers m, n, and h. Its altitude has length $2mnh$. (It is scaled too!)

Write these three rationals as fractions with with a common denominator d:

$$m = \frac{M}{d}, \quad n = \frac{N}{d}, \quad h = \frac{H}{d}.$$

Scaling by d^3 shows that our Heronian triangle is similar to one with integer side lengths

$$N(M^2 + H^2)$$
$$M(N^2 + H^2)$$
$$(M+N)(MN - H^2)$$

for three integers M, N, and H.

The altitude of this triangle is $2mnhd^3 = 2MNH$ and so the area of the triangle is the integer

$$\frac{1}{2} \cdot (M+N)(MN - H^2) \cdot 2MNH$$
$$= MNH(M+N)(MN - H^2)$$

This is the claimed result (with a slight change of notation.)

Example: Earlier we pasted together the 6-8-10 and 8-15-17 right triangles to create the 10-17-21 integer Heronian triangle.

Can we find integers m, n, and h so that

$$10 = n(m^2 + h^2)$$
$$17 = m(n^2 + h^2) \qquad ?$$
$$21 = (m+n)(mn - h^2)$$

No! Since 17 is prime, we must have $m = 1$. Thus, we either have $n = 4$, $h = 1$ or $n = 1$, $h = 4$. But neither of these work for the first equation.

However, the 10-17-21 triangle is similar to the 20-34-42 triangle, which does arise by choosing $m = 2$, $n = 4$, $h = 1$.

SCALING INTEGER HERONIAN TRIANGLES

If a triangle has sides of lengths a, b, c and area A and we scale each side length by a factor k, then the $ka - kb - kc$ triangle has area $k^2 A$.

Result 12: If each side of an integer Heronian triangle is scaled by an integer, the result is another integer Heronian triangle.

It is not immediately obvious that for integer Heronian triangles, the converse is true!

Result 13: If an integer Heronian triangle has integer side lengths ka, kb, kc and area A, then the $a-b-c$ triangle has integer area and so is also integer Heronian.

Reason: The $a-b-c$ triangle has area given by

$$B=\frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$

So $B=\dfrac{\sqrt{N}}{4}$ for some integer N.

We also know that scaling this $a-b-c$ triangle by a factor k gives us a triangle of integer area A. So $A=k^2B$.

Thus we have $\sqrt{N}=\dfrac{4A}{k^2}$ is a rational number. It must be the case then that the integer N is a perfect square and $\dfrac{4A}{k^2}$ is an integer.

If k is odd then it must be that $\dfrac{A}{k^2}$ is an integer and so $B=\dfrac{\sqrt{N}}{4}=\dfrac{A}{k^2}$ is a whole number, as hoped.

The case k is even, however, is delicate.

Let's start by establishing the following:

If p, q, r are integers such that the $2p-2q-2r$ triangle has integer area C, then C is a multiple of four, and the $p-q-r$ triangle has integer area $\dfrac{C}{4}$.

We have

$$C^2=(p+q+r)(-p+q+r)(p-q+r)(p+q-r)$$

As per the reasoning of result 3, either each term in this product is even or each is odd.

In the first case, C^2 is a multiple of 16 and the claim follows. We'll show that that the latter case is impossible.

Suppose $p+q+r=2k+1$ for some integer k. Then

$$C^2=(2k+1)(2k+1-2a)(2k+1-2b)(2k+1-2c)$$

Multiplying this out grouping multiples of 4 along the way gives

$$C^2=4(something)+3.$$

But no number squared is 3 more than a multiple of four. (Try squaring an even number and an odd number.) So, it is not possible that $p+q+r$ is odd.

So C, the area of the $2p-2q-2r$ triangle, is indeed a multiple of 4. The half-scale $p-q-r$ triangle has area $\dfrac{C}{4}$, and this is sure to an integer.

To summarise:

We know that if the three sides lengths of an integer Heronian triangle share a common odd factor, then we can divide through by that factor and obtain another integer Heronian triangle. The same is true if each side length shares a factor of common factor of 2.

By repeatedly dividing by 2s and odd numbers, we can thus divide the side lengths by any common factor we choose to create a smaller integer Heronian triangle.

The claim of result 13 is thus true.

Result 14: The area of any integer Heronian triangle is even.

Reason: Suppose the triangle has sides of lengths a, b, c, semi-perimeter s, and area A. We want to establish that A is even.

Now $A^2 = s(s-a)(s-b)(s-c)$.

Could it be that each term in this product is an odd number? If so, s and $s-a$ each odd means a is even. Similarly, b and c are even.

But in the proof of result 13 we established that this means that A is a multiple of 4 and not the product of four odd numbers after all.

It must be that at least one of the terms s, $s-a$, $s-b$, $s-c$ is even, making A^2, and hence A, even.

Result 15: The area of any integer Heronian triangle is a multiple of three.

Reason: Suppose the triangle has sides of lengths a, b, c and area A. We want to establish that A is a multiple of three.

We know
$16A^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$
with $a+b+c$ even.

Suppose $a+b+c$ is not a multiple of three. (If it is, A must be a multiple of three, and we're done.)

Then A is either 2 more or 2 less than a multiple of 6.

Consider the case $a+b+c = 6k+2$ for some integer k.

Then
$16A^2 = (6k+2)(6k+2-2a)(6k+2-2b)(6k+2-2c)$
giving

$$A^2 = (3k+1)(3k+1-a)(3k+1-b)(3k+1-c)$$

Expanding we see that

$$A^2 = 3(something) + (1-a)(1-b)(1-c)$$

Using arithmetic mod 3 we have

$$A^2 \equiv (1-a)(1-b)(1-c)$$

Also, $a+b+c \equiv 2$ mod 3 and so we have that, mod 3,

$(a,b,c) = (0,0,2)$ or some permutation of this. (And this gives $A^2 \equiv 2$)

or $(a,b,c) = (1,2,2)$ or some permutation of this. (And this gives $A^2 \equiv 0$)

or $(a,b,c) = (0,1,1)$ or some permutation of this. (And this gives $A^2 \equiv 0$)

It is impossible for a square number to be congruent to 2 mod 3, so we have that $A^2 \equiv 0$. Thus A^2, and hence A, is a multiple of 3.

Now consider the alternative case $a+b+c = 6k-2$ for some integer k.

Then one checks in a similar manner that

$$A^2 \equiv (1+a)(1+b)(1+c)$$

mod 3. For $a+b+c \equiv 1$ we have only the possibilities that (a,b,c) is $(0,0,1)$, $(1,2,1)$, or $(0,2,2)$, or some permutation of these. Again, each possibility gives $A^2 \equiv 2$ or $A^2 \equiv 0$.
As before, we conclude that A is a multiple of three.

Corollary 16: The area of any integer Heronian triangle is a multiple of six.

Question: Where is a (full and correct) proof of the fact that the area of an integer Heronian triangle is a multiple of six published?

EXTRA THINKING

Apparently Brahmagupta noticed that Heron's formula for the area of a triangle is asymmetrical and, to remedy this, imagined a triangle to be a quadrilateral with a fourth side of length $d = 0$. This then provides a lovely symmetrical presentation of Heron's formula.

$$A = \sqrt{\frac{a+b+c-d}{2} \cdot \frac{a+b-c+d}{2} \cdot \frac{a-b+c+d}{2} \cdot \frac{-a+b+c+d}{2}}$$

In terms of the semi-perimeter, this reads

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

But this now this begs the question:

Is this a formula for the area of a general quadrilateral with sides of lengths a, b, c, d?

If not, this is the area formula for which special class of quadrilaterals? (And, beyond this class, what then is the most general area formula of all for quadrilaterals?)

RESEARCH CORNER

What other claims about Heronian triangles are stated on websites whose proofs are mighty tricky to track down? Can you provide proofs of the claims using only tools from high-school mathematics? (Well, in this essay I did bring in arithmetic modulo three in one of my proofs. But I could have phrased that passage entirely in terms of remainders 0, 1, and 2 upon division by three.)

Comment: The answer to the "Extra Thinking" question can be readily found on the internet. The proofs of "Brahmagupta's Formula" and "Bretschneider's Formula" are readily found too.

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