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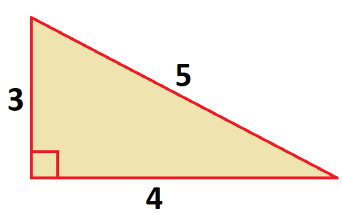
WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JUNE 2021

THIS MONTHS' PUZZLER:

The classic 3-4-5 integer right triangle has area 6, an integer.



a) Does every right triangle with integer side lengths have integer area?

b) The 3-4-5 triangle has side lengths three consecutive integers. Is there another example of a triangle, not necessarily right, with three consecutive integers for its sides lengths and having integer area?

HERONIAN TRIANGLES

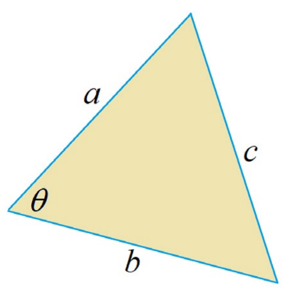
Hero of Alexandria (ca. 10 CE – 70 CE) was a Greco-Egyptian mathematician and engineer best remembered in mathematics for his formula for the area of a triangle based solely on its three side lengths.

Heron's Formula:
If a triangle has side lengths a, b, and c, then its area A is given by

$$A=\sqrt{\frac{a+b+c}{2}\cdot\frac{-a+b+c}{2}\cdot\frac{a-b+c}{2}\cdot\frac{a+b-c}{2}}$$

Challenge: For a horrendous exercise in algebra, try proving Heron's formula as follows.

Let θ be the angle between the sides of lengths a and b.



The area formula

$$A=\frac{1}{2}ab\sin\theta$$

gives a formula for $\sin\theta$ in terms of a, b, and A. The law of cosines

$$c^2=a^2+b^2-2ab\cos\theta$$

gives a formula for $\cos\theta$ in terms of a, b, and c.

The equation $\cos^2\theta+\sin^2\theta=1$ thus supplies us with an equation solely in terms of a, b, c, and A. Show that solving for A gives Heron's formula.

Challenge: If $a^2+b^2=c^2$, show that Heron's formula reduces to $A=\frac{1}{2}ab$, as it should!

Hint: First compute $((a+b)+c)((a+b)-c)$ and $(c+(a-b))(c-(a-b))$.

A triangle with three integer side lengths is today called **Heronian** if its area A has integer value too.

For example, the 3-4-5 triangle is Heronian. So too is the 5-29-30 triangle of area $A=72$, and the 13-14-15 triangle of area $A=84$, and the 7-15-20 triangle of area $A=42$.

Result: Every right triangle with integer side lengths is Heronian.

Reason: Suppose a right triangle has side lengths a, b, c with $a^2+b^2=c^2$. Its area is $A=\frac{ab}{2}$. For this to be an integer, we need to be sure that at least one of a or b is even.

Could they both be odd?

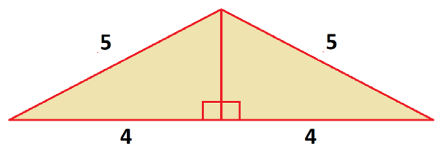
If they are, then both a^2 and b^2 being odd, makes c^2 an even number. But if c^2 is even, so too must be c. So $c=2k$ for some value k. Since we are presuming a and b are each odd, they have the forms $a=2n+1$ and $b=2m+1$ for some value m and n.

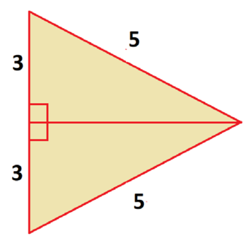
The equation $a^2+b^2=c^2$ then reads: $4m^2+4m+1+4n^2+4n+1=4k^2$.

But this is absurd. The left side is a number two more than a multiple of four and it equals a number on the right that is exactly a multiple of four.

It cannot be the case then that both a and b are odd. At least one is even and we can thus be sure that $A = \frac{ab}{2}$ is an integer.

Challenge: One can paste together two copies of an integer right triangle along a common side length to create a Heronian isosceles triangle.





Can you find two non-congruent integer right triangles that can be pasted together to create a scalene Heronian triangle? Can you find infinitely many such non-congruent pairs?

Challenge: The 5-12-13 integer right triangle has perimeter 30 and area 30. Prove that there is only one more integer right triangle with area and perimeter having the same numerical value.

Comment: There are only five Heronian triangles with area equal to perimeter: the two right triangles above, and the 6-25-29, 7-15-20, and 9-10-17 triangles.

HERONIAN TRIANGLES WITH THREE CONSECUTIVE INTGER SIDE LENGTHS

The Heronian triangles, the 3-4-5 and the 13-14-15 triangles, have side lengths three consecutive integers.

There are infinitely many such triangles, and we can describe them all!

Suppose a triangle with side lengths $k-1$, k, $k+1$ is Heronian. What can we say about the value k?

The triangle's area A is given by

$$A = \sqrt{\frac{3k}{2} \cdot \frac{k+2}{2} \cdot \frac{k}{2} \cdot \frac{k-2}{2}}$$

and is an integer.

Playing with this formula gives

$$(4A)^2 = 3k^2\left(k^2-4\right).$$

The left side is an even number, which informs us that k must be even. Write $k = 2a$ for some value a. We see

$$16A^2 = 3 \cdot 4a^2\left(4a^2-4\right)$$

or

$$A^2 = 3a^2\left(a^2-1\right).$$

Now the left side is a square number, so the right side must be a square number too. But it has a lone prime number 3 sitting in it. What does this then say about the factors of 3 that appear in the remaining terms of the equation, A^2 and a^2 and a^2-1?

Now any factor of 3 that appears in square number must do so an even number of times. (Squaring a number doubles the count of appearances of each of its prime factors.) So 3 appears an even number of times as a factor of A^2, the left side of the equation.

Thus 3 must appear an even number of times as a factor in the right side of the equation as well. It appears once at the front, it will appear an even number of times in a^2, so it must be the case that it appears an odd number of times as a factor of a^2-1. In particular, it appears at least once.

We have: a^2-1 is a multiple of 3.
Write $a^2-1=3m$.

So now our equation reads:

$$A^2=9a^2m$$

with any factors of 3 that occur in m doing so an even number of times.

Actually, since A^2, 9, and a^2 are each square numbers, any prime number that occurs as a factor of m must do so an even number of times. This means that m is itself a square number too!

Write $m=b^2$.

So, where are we?

We started with a triangle with side lengths $k-1$, k, and $k+1$, and assumed A is an integer.

This led us to say that $k=2a$ for some integer a with the property that $a^2-1=3b^2$ for some other integer b.

Backwards?

Suppose we find a pair if integers a and b satisfying

$$a^2-1=3b^2.$$

Then we can set $k=2a$ and see that the triangle with sides $k-1$, k, $k+1$ has area

$$A=\sqrt{\frac{3k}{2}\cdot\frac{k+2}{2}\cdot\frac{k}{2}\cdot\frac{k-2}{2}}$$

which equals

$$\begin{aligned}A&=\sqrt{3a^2\left(a^2-1\right)}\\&=\sqrt{3a^2\cdot 3b^2}\\&=3ab\end{aligned}$$

an integer!

Every pair of integers (a,b) satisfying

$$a^2-1=3b^2$$

gives us a Heronian triangle with side lengths $2a-1$, $2a$, $2a+1$.

And, conversely, every Herononian triangle with three consecutive integer side lengths is of this form.

So, to find Heronian triangles of the form we seek we need to find integers a and b which satisfy the equation $a^2-1=3b^2$.

SOLVING THE EQUATION

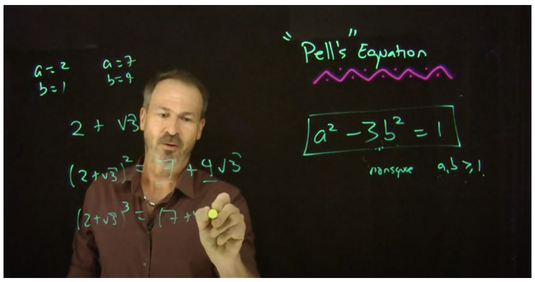
Equations of the form $a^2-nb^2=1$ for a given number n are famous in the history of mathematics. They are (mistakenly) known as "Pell's equation."

[John Pell was a 1600s English mathematician who wrote a revised edition of a book by Thomas Branker, which was a translation of a book by Johann Rahn, who described in it a method by Willian Brouncker to solve such equations! World famous 18th-century mathematician Leonard Euler read Pell's tome and mistakenly attributed the proof to Pell, and the name "Pell's equation" stuck. However, the full method for solving these equations was actually developed some 500 years earlier in India by Bhāskara II (ca. 1150), based on ideas from Brahmagupta (ca. 590).]

The particular version of Pell's equation we have,

$$a^2 - 3b^2 = 1,$$

arose in a recent video I made in solving a probability problem. In that video I describe and explain the complete mathematics for solving the equation, that is, for finding all counting numbers a and b that fit it.



https://youtu.be/sIpprPNRzRc

The solutions arise from simply looking at the powers of $2+\sqrt{3}$. (Whoa!)

$$2+\sqrt{3} = 2 + 1\cdot\sqrt{3}$$

and $a=2$, $b=1$ is a solution.

$$\left(2+\sqrt{3}\right)^2 = 7 + 4\cdot\sqrt{3}$$

and $a=7$, $b=4$ is a solution.

$$\left(2+\sqrt{3}\right)^3 = 26 + 15\cdot\sqrt{3}$$

and $a=26$, $b=15$ is a solution.

$$\left(2+\sqrt{3}\right)^4 = 97 + 56\cdot\sqrt{3}$$

and $a=97$, $b=56$ is a solution.

$$\left(2+\sqrt{3}\right)^5 = 362 + 209\cdot\sqrt{3}$$

and $a=362$, $b=209$ is a solution.

and so on.

Every solution to $a^2 - 3b^2 = 1$ will appear in this list. (Do look at the video!)

Each value of a gives a Heronian triangle with sides $2a-1$, $2a$, $2a+1$, and the area of such a triangle is given by $A = 3ab$.

a:	2	7	26	97	362	...
b:	1	4	15	56	209	...
k = 2a:	4	14	52	194	724	...
A = 3ab:	6	84	1170	16926	226974	...

This the complete list of Heronian triangles that are "almost equilateral."

Challenge: Is the 3-4-5 Heronian triangle the only one with three consecutive integer side lengths that is also a right triangle?

Challenge: Consider the sequence

$$k_n : 4,\ 14,\ 52,\ 194,\ 724,\ \ldots$$

If we set $k_0 = 2$, show that

$$k_n = \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n.$$

[How do the values of $\left(2+\sqrt{3}\right)^n$ and $\left(2-\sqrt{3}\right)^n$ compare?]

Show that we have $k_{n+2} = 4k_{n+1} - k_n$.

RESEARCH CORNER

The 3-4-5 Heronian triangle has area A and perimeter P satisfying $P = 2A$. Are there any more examples of such Heronian triangles?

How about ones with $A = 2P$ or $P = 3A$, and so on?

James Tanton
tanton.math@gmail.com