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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

APRIL 2018

THIS MONTH'S PUZZLER:
We have

$$75^2+15^2=51^2+57^2$$

$$27^2+96^2=69^2+72^2$$

$$17^2+89^2=98^2+71^2$$

Find other equal sums of two squares that can be read forward and backwards!

SUMS OF TWO SQUARES

Some numbers can be written as the sum of two squares (for example, $10=1^2+3^2$), other numbers are not the sum of two squares (for example, $23\neq a^2+b^2$ for any integers a and b), and some numbers can be written as a sum of two squares in more than one non-trivial way (for example, $83=2^2+9^2=6^2+7^2$ and $8450=13^2+91^2=23^2+89^2$ $=35^2+85^2=47^2+79^2$). People usually consider $0=0^2$ to be a square number and so consider writing $25=0^2+5^2$ as a valid representation of 25, for example, as a sum of two squares.

The opening puzzle is about numbers that can be written as a sum of two squares in at least two non-trivial ways, with the extra property of having a curious base-ten reverse symmetry.

Something Cute

Suppose $M = uv$ with u and v both odd or both even and $u \geq v$. Then $\frac{u+v}{2}$ and $\frac{u-v}{2}$ are both integers and we have

$$M = uv = \left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2.$$

Now look for two different factorizations (with factors the same parity) of the same number M. Doing so gives you examples of numbers that are sums of two squares in two different ways.

From $15 = 5 \cdot 3$, for example, we see $15 = 4^2 - 1^2$, and from $15 = 15 \cdot 1$ we see $15 = 8^2 - 7^2$. Thus

$$4^2 - 1^2 = 8^2 - 7^2.$$

Rearranging gives

$$4^2 + 7^2 = 8^2 + 1^2$$

showing that 65 is a as a sum of two squares two different ways.

Writing $45 = 45 \cdot 1 = 15 \cdot 3 = 9 \cdot 5$ gives the examples

$$6^2 + 7^2 = 2^2 + 9^2$$

$$7^2 + 22^2 = 2^2 + 23^2$$

$$9^2 + 22^2 = 6^2 + 23^2.$$

From $24 = 12 \cdot 2 = 4 \cdot 6$ we get $1^2 + 7^2 = 5^2 + 5^2$.

> **Exercise:** Show that if M is an odd composite number or a multiple of eight greater than eight, then it has at least two different factorizations $M = uv$ with u and v of the same parity.

REVERSIBLE SUMS OF TWO SQUARES

Every two-digit number can be written in the form $10a + b$ with a and b single digits. The opening puzzle is thus looking for single digits a, b, c, and d satisfying

$$(10a+b)^2 + (10c+d)^2$$
$$= (10d+c)^2 + (10b+a)^2$$

(at least in the hunt for two-digit reversible examples). Algebra shows that this is equivalent to

$$a^2 + c^2 = b^2 + d^2$$

Thus to find reversible two-digit sums of two squares all we need do is find single-digit examples of numbers that can be written as a sum of two squares in two different ways.

For example, from $2^2 + 9^2 = 7^2 + 6^2$ read $a = 2$, $b = 7$, $c = 9$, $d = 6$ to get

$$27^2 + 96^2 = 69^2 + 72^2.$$

The variation $2^2 + 9^2 = 6^2 + 7^2$ gives

$$26^2 + 97^2 = 79^2 + 62^2.$$

Alternatively, just look at the sequence of digits 2, 9, 6, 7 you see in $2^2 + 9^2 = 6^2 + 7^2$ and write them forwards (red) and then backwards (blue) to create the two-digit equation.

$$27^2 + 96^2 = 69^2 + 72^2$$

GOING FURTHER

One can check that

$$(ax+by)^2 + (cx+dy)^2$$
$$= (dx+cy)^2 + (bx+ay)^2$$

holds precisely when

$$a^2 + c^2 = b^2 + d^2$$

for any pair of values $x \neq y$. For our two-digit examples in base-ten we used $x = 10$ and $y = 1$. One can find reversible sums of two squares in any base b one likes simply by using $x = b$ and $y = 1$ (provided you can find single-digit pairs of sums).

Back in base-ten, we can choose x to be any power of ten and $y=1$ to get a whole host of curious reversible sums of squares from a single starting point. For example, from $2^2+9^2=6^2+7^2$ we get

$$27^2+96^2=69^2+72^2$$
$$207^2+906^2=609^2+702^2$$
$$2007^2+9006^2=6009^2+7002^2$$
$$20007^2+90006^2=60009^2+70002^2$$

and so on.

From $13^2+91^2=23^2+89^2$ and choosing $x=100$ and $y=1$ gives

$$1389^2+9123^2=2391^2+8913^2$$

Alas, this is not a reversible example, but it is nonetheless cool.

Choosing different values for x and y generates, from a single example of two equal sums of two-squares an infinitude of additional examples.

RESEARCH CORNER

Are there three-digit examples of reversible sums of two-squares? Higher digit examples?

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AN ILL-DEFINED RESEARCH IDEA

This essay is motived by an arithmetic example I saw in article. It suggested having young students explain why 12×42 equals 21×24. I realized that if one rewrites this equation, it is "reversible."

$$12\times 42=24\times 21$$

This led me to examine how one might construct other examples of "reversible equal products."

Exercise: Show that $(10a+b)(10c+d)$ $=(10d+c)(10b+a)$ if, and only if, $ac=bd$.

But why stop at multiplication?

Exercise: Show that $(10a+b)+(10c+d)$ $=(10d+c)+(10b+a)$ if, and only if, $a+c=b+d$.

Exercise: Show that $(10a+b)-(10c+d)$ $=(10d+c)-(10b+a)$ if, and only if, $a-c=b-d$.

An exploration of other binary operations that might carry from single-digit results to double-digit ones led me to examine $x*y=x^2+y^2$ and the results of this essay.

Vague Research Question: Develop a theory about binary operations for which $a*c=b*d$ holding forces

$$F(a,b)*F(c,d)=F(d,c)*F(b.a)$$

to hold too for a given class of functions F. (We've been working with the linear functions, $F(a,b)=10a+b$ and $F(a,b)=ax+by$.)