SOME THOUGHTS ON PARTITIONS

This material is adapted from a forth-coming book

INTRODUCTION:

A partition of a counting number N is an expression that represents N as a sum of counting numbers. For example, there are eight partitions of the number 4 if order is considered important:

$$4 \quad 3+1 \quad 1+3 \quad 2+2$$
$$2+1+1 \quad 1+2+1 \quad 1+1+2 \quad 1+1+1+1$$

There are just five partitions of the number 4 if order is not considered important:

$$4 \quad 3+1 \quad 2+2 \quad 2+1+1 \quad 1+1+1+1+1$$

QUESTION 1: How many ordered partitions are there of the numbers 1 through 6. Any patterns?

QUESTION 2: How many unordered partitions are there of the numbers 1 through 6. Any patterns? Are these numbers always prime?

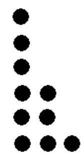
One can place all sorts of restrictions on the types of partitions one wishes to count. For example, there are eight partitions of the number 10 with exactly three terms, order not important:

$$8+1+1 \quad 7+2+1 \quad 6+3+1 \quad 6+2+2$$
$$5+4+1 \quad 5+3+2 \quad 4+4+2 \quad 4+3+3$$

QUESTION 3: Show that there are eight ways to partition the number 10 using only the numbers 1, 2, and 3, order immaterial, using at least one 3.

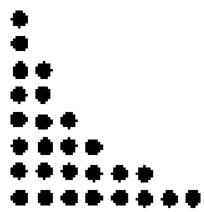
QUESTION 4: Let $A(N)$ be the number of ways to partition N as a sum of exactly three terms, and let $B(N)$ be the number of partitions of N with largest term 3. (Order immaterial in both cases.) Question 3 shows $A(10) = 8 = B(10)$. Prove that $A(N) = B(N)$ always.

Any partition of a number can be represented as columns of dots. For example, the partition $6+3+1$ of the number 10 can be depicted:

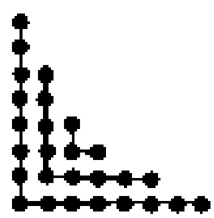


Comment: If order is not considered important, then it is customary to place the terms in order from biggest to smallest. From now on we shall assume that for all our partitions order is indeed immaterial.

If one looks at this diagram sideways, we see the alternative partition $3+2+2+1+1+1$ of the number 10. These two partitions are said to be "conjugate" partitions. Some partitions are "self-conjugate." For example, the partition $8+6+4+3+2+2+1+1$ of the number 27 is self-conjugate.



Notice that the diagram of any self-conjugate partition of a number can seen as a series of nested L-shapes, each L with an odd number of dots.



Also, no two L shapes can contain the same number of odd dots.

This shows that each self-conjugate partition of a number gives rise to a partition into distinct odd terms. (We see that 27 can be written $15+9+3$.) And conversely, any partition into distinct odd terms can be seen as arising from a self-conjugate partition. We have:

The count of self-conjugate partitions of a number equals the count of partitions into distinct odd numbers.

ONE OF EULER'S REMARKABLE RESULTS:

In 1740 French mathematician Philippe Naudé sent a letter to Leonhard Euler asking him about the number of ways a positive integer could be written as a sum of distinct positive integers (order immaterial). Euler examined the problem and discovered something truly remarkable.

Let $D(n)$ be the number of ways to write n as a sum of distinct positive integers. For instance, $D(6)=4$ because we can write six as 6 or 5+1 or 4+2 or 3+2+1.

EXERCISE: Find $D(2), D(5)$ and $D(13)$.

Let $O(n)$ be the number of ways to write n as a sum of odd integers. For example, $O(6)=4$ because we can write six as 5+1 or 3+3 or 3+1+1+1 or 1+1+1+1+1+1..

EXERCISE: Find $O(2), O(5)$ and $O(13)$.

EULER: $O(N)=D(N)$ always

Euler's Proof: What would happen if one expanded the infinite quantity

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots ?$$

By selecting "1" from each set of parentheses we see that we would obtain the constant term 1. By selecting x and then 1s thereafter, we see that the term x will arise. We also see that the term x^2 will arise by selecting 1, then x^2 and then 1s thereafter. Skipping ahead, how will the term x^6 arise? We see that this arises in more than one way: As x^6 and 1s elsewhere; as x and x^5 and 1s elsewhere; as x^2 and x^4 and 1s; as x^3 and x^2 and x and 1s. There are 4 ways we would see x^6 appear and they each match a partition of 6 into distinct numbers: 6, $5+1$, $4+2$, $3+2+1$. In general, the coefficient of x^N in this infinite product when expanded out is $D(N)$. We have:

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots \ = 1+x+x^2+2x^3+2x^4+3x^5+4x^6+\cdots$$

Now consider $\frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdots$.

Euler knew the famous geometric series that we teach our precalculus students:

$$\frac{1}{1-z}=1+z+z^2+z^3+\cdots$$

and using it we see that this second infinite product can be rewritten:

$$\begin{aligned}&\left(1+x+x^{1+1}+x^{1+1+1}+\cdots\right)\\&\times\left(1+x^3+x^{3+3}+x^{3+3+3}+\cdots\right)\\&\times\left(1+x^5+x^{5+5}+x^{5+5+5}+\cdots\right)\\&\times\cdots\end{aligned}$$

What do we obtain if we expand this out? By selecting 1 from each set of parentheses there is certainly a constant term of 1. There will also be a single x term, and a single x^2 term (from selecting x^{1+1}).There are four ways x^6 arises: as x^{3+3}; as $x^5 \cdot x^1$; as $x^3 \cdot x^{1+1+1}$; as $x^{1+1+1+1+1+1}$. In fact, we see that x^N arises from the odd partitions of N. The coefficient of x^N is $O(N)$.

If we can prove that the two different infinite products we examined are actually equal, then it would follow that their expanded versions are equal. This will give $O(N)=D(N)$ always. Okay. Here goes!

$$\begin{aligned}&\frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdots\cdot\\&=\frac{1}{1-x}\cdot\frac{1-x^2}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{1-x^4}{1-x^4}\cdot\frac{1}{1-x^5}\cdot\frac{1-x^6}{1-x^6}\cdot\frac{1}{1-x^7}\cdots\cdot\\&=\frac{1}{1-x}\cdot\frac{(1-x)(1+x)}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{(1-x^2)(1+x^2)}{1-x^4}\cdot\frac{1}{1-x^5}\cdot\frac{(1-x^3)(1+x^3)}{1-x^6}\cdot\frac{1}{1-x^7}\cdots\cdot\end{aligned}$$

Cross out common terms and we are left with: $(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$. DONE!

Comment: Euler was not at all bashful about playing with the infinite!

Another Remarkable Result: EULER'S PENTAGONAL THEOREM

Let $P(N)$ denote the count of unordered partitions of a number N subject to no restrictions. (These are often called the partition numbers.) The first ten values of $P(N)$ are:

N	1	2	3	4	5	6	7	8	9	10
P(N)	1	2	3	5	7	11	15	22	30	42

and these numbers grow rapidly without bound: $P(100)=190,569,292$ and $P(1000)\approx 2.4\times 10^{31}$.

Leonhard Euler made the most peculiar and astounding discovery in partition theory. One can learn more of this mathematics in [HARDY and WRIGHT] and in [TATTERSALL], for instance. Robert Young in [YOUNG] presents a translated passage from Euler's memoir that describes Euler's process in discovering and establishing the piece of mathematics I am about to describe.

WARNING: This is demanding reading!

We saw in the opening section that the infinite product

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$

when expanded out "generates" the numbers $D(N)$, the count of ways to write N as a sum of distinct terms, order immaterial. (The number $D(N)$ appears as the coefficient of x^N .) We have:

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots = 1+x+x^2+2x^3+2x^4+3x^5+4x^6+\cdots$$

Euler wondered about the related infinite product:

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$$

When expanded it appears as:

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+\cdots$$

The signs of terms seem to alternate in pairs and the exponents that appear are:

0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, …

Euler observed taking the difference between consecutive terms yields the sequence

1, $\underline{1}$, 3, $\underline{2}$, 5, $\underline{3}$, 7, $\underline{4}$, 9, $\underline{5}$, 11, …

which is the sequence of counting numbers 1, 2, 3, 4, … intertwined with the sequence of odd numbers 1, 3, 5, 7, 9, …., and this observation allows us to generate additional terms of the sequence of exponents with some ease. As Euler observed, one can use an induction argument to verify these claims but such a route doesn't offer clarity on an intuitive level. In what way does the infinite product $(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$ connect with partitions?

Getting a Handle on this Infinite Product

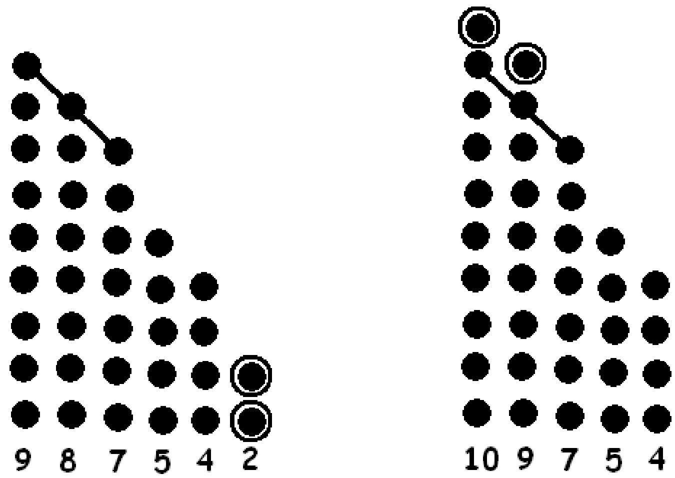
In expanding $(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$ each partition $a+b+c+\cdots+z$ of N into distinct terms contributes once to the coefficient of x^N (from selecting $+x^a$ and $+x^b$ and $+x^c$ and so on, and 1s elsewhere among the parentheses). In expanding $(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$ the partition $a+b+c+\cdots+z$ contributes $+1$ to the coefficient of x^N if it involves an even number of terms, and -1 if it involves an odd number of terms.

Let $D_{even}(N)$ and $D_{odd}(N)$ denote, respectively, the count of ways to write N as a sum of an even/odd number of distinct terms. We see:

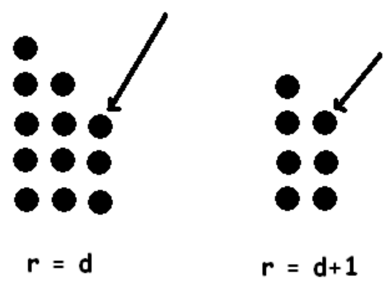
$$(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$$
$$=1+(D_{even}(1)-D_{odd}(1))x+(D_{even}(2)-D_{odd}(2))x^2+\cdots+(D_{even}(N)-D_{odd}(N))x^N+\cdots$$

In 1881 mathematician Fabian Franklin noted that a partition of a number into an even count of distinct parts often matches a partition of that number into an odd count of distinct parts. We'll deviate from Euler's approach at this point (and return to it later on) and explain the work of Franklin. It is clearly relevant.

Any diagram of a partition possesses a rightmost column of dots and, from the top left dot, a southwest diagonal of dots at a 45 degree slope. Let r denote the number of dots in its rightmost column and d the number of dots in the diagonal. (The left picture below has $r=2$ and $d=3$.) If r is smaller than d, then moving the r dots in the rightmost column to lay them on top of the diagonal produces a partition still with distinct terms, but with one less term overall. If r is bigger than $d+1$, then moving dots from the diagonal and making a new rightmost column also creates a new partition still with distinct terms but now with one more term overall. In each case, these transformations convert a partition with an even number of distinct terms into one with an odd number of distinct terms, and vice versa.



The first transformation also works for $r = d$ as long as the dot at the top of the rightmost column is not part of the diagonal. (The diagram on the left below illustrates this exceptional case.) The second transformation also works for $r = d + 1$ as long as, again, no dot lies both in the rightmost column and the diagonal. (The right diagram below illustrates this.)



If the first exceptional case occurs, then N must be a number of the form:

$$\frac{1}{2}d \times \left(d + \left(d-1\right) + d\right) = \frac{d\left(3d-1\right)}{2}$$

(the dots form half a rectangle) and there is this one partition of N into d distinct terms we cannot match with a different partition. All other partitions match in pairs.

If the second exceptional case occurs, then N must be a number of the form:

$$\frac{1}{2}d \times \left(d + \left(d+1\right) + d\right) = \frac{d\left(3d+1\right)}{2}$$

and there is this one partition of N into d distinct terms we cannot match with a different partition. All other partitions match in pairs.

For all other values of N (that is, those not of the form $\frac{d(3d \pm 1)}{2}$ for some integer d), all even and odd partitions of N match perfectly in pairs.

Thus we have:

$$D_{even}(N) - D_{odd}(N) = 0 \text{ if } N \text{ is not of the form } \frac{d(3d \pm 1)}{2}.$$

$$D_{even}(N) - D_{odd}(N) = 1 \text{ if } N \text{ is of the form } \frac{d(3d \pm 1)}{2} \text{ with } d \text{ even.}$$

$$D_{even}(N) - D_{odd}(N) = -1 \text{ if } N \text{ is of the form } \frac{d(3d \pm 1)}{2} \text{ with } d \text{ odd.}$$

Plugging in $d = 1, 2, 3, 4, \ldots$ notice that the formulas $\frac{d(3d \pm 1)}{2}$ yield the sequence $1, 2, 5, 7, 12, 15, \ldots$ with $d = 1$ giving 1 and 2 (and d odd here corresponds to a coefficient of $+1$); $d = 2$ giving the terms 5 and 7 (and d even here corresponds to a coefficient of -1); $d = 3$ giving the terms 12 and 15; and so on. From

$$\begin{aligned}&(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots \\ &\quad = 1 + (D_{even}(1) - D_{odd}(1))x + (D_{even}(2) - D_{odd}(2))x^2 + (D_{even}(3) - D_{odd}(3))x^3 + \cdots\end{aligned}$$

we now see that

$$\begin{aligned}&(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots \\ &\quad = 1 + (-1)x + (-1)x^2 + (0)x^3 + (0)x^4 + (+1)x^5 + (0)x^6 + (+1)x^7 + \cdots \\ &\quad = 1 - x - x^2 + x^5 + x^7 - \cdots\end{aligned}$$

At last we understand the alternating pairs of signs and the numbers appearing as exponents.

Connection to General Partitions

Recall that $P(N)$ denotes the number of ways to write N as a sum of terms with no restrictions (and order is considered irrelevant).

We have seen that $(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$ generates those partitions in which each number is used at most once (the partitions into distinct terms).

The product $\left(1+x^{1}+x^{1+1}\right)\left(1+x^{2}+x^{2+2}\right)\left(1+x^{3}+x^{3+3}\right)\left(1+x^{4}+x^{4+4}\right)\cdots$
generates those partitions that use each number at most twice, and the product
$\left(1+x^{1}+x^{1+1}+x^{1+1+1}\right)\left(1+x^{2}+x^{2+2}+x^{2+2+2}\right)\left(1+x^{3}+x^{3+3}+x^{3+3+3}\right)\left(1+x^{4}+x^{4+4}+x^{4+4+4}\right)\cdots$
those that use each number at most thrice, and so on.

The infinite product:

$$\begin{aligned}&\left(1+x^{1}+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+\cdots\right)\\&\quad\times\left(1+x^{2}+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+\cdots\right)\\&\quad\times\left(1+x^{3}+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+\cdots\right)\\&\quad\times\left(1+x^{4}+x^{4+4}+x^{4+4+4}+x^{4+4+4+4}+\cdots\right)\\&\quad\times\cdots\end{aligned}$$

generates the partitions with no limit on the count of times each particular number is used. That is, it generates the numbers $P(N)$. [Check this: Show that there are seven ways to produce the term x^{5}, each way corresponding to a partition of 5.]

We have then that this infinite product equals

$$1+P(1)x+P(2)x^{2}+P(3)x^{3}+P(4)x^{4}+\cdots$$

The geometric series formula shows that the infinite product also equals:

$$\frac{1}{1-x}\cdot\frac{1}{1-x^{2}}\cdot\frac{1}{1-x^{3}}\cdot\cdots$$

which tells us that multiplying it by $(1-x)\left(1-x^{2}\right)\left(1-x^{3}\right)\left(1-x^{4}\right)\cdots$ must yield the value one. Whoa! This is saying that multiplying

$$1+P(1)x+P(2)x^{2}+P(3)x^{3}+P(4)x^{4}+\cdots$$

by

$$1-x-x^{2}+x^{5}+x^{7}-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+\cdots$$

gives one. Let’s write this out:

$$1=\left(1+P(1)x+P(2)x^2+P(3)x^3+P(4)x^4+\cdots\right)\left(1-x-x^2+x^5+x^7-x^{12}-x^{15}+\cdots\right)$$

$$\begin{aligned}=1+P(1)x+P(2)x^2+P(3)x^3+P(4)x^4+P(5)x^5+P(6)x^6+P(7)x^7+\cdots\\ -x-P(1)x^2-P(2)x^3-P(3)x^4-P(4)x^5-P(5)x^6-P(6)x^7-\cdots\\ -x^2-P(1)x^3-P(2)x^4-P(3)x^5-P(4)x^6-P(5)x^7-\cdots\\ +x^5+P(1)x^6+P(2)x^7+\cdots\\ +x^7+\cdots\\ -\cdots\end{aligned}$$

Thus we can see:

$$P(1)-1=0$$
$$P(2)-P(1)-1=0$$
$$P(3)-P(2)-P(1)=0$$
$$P(4)-P(3)-P(2)=0$$
$$P(5)-P(4)-P(3)+1=0$$
$$P(6)-P(5)-P(4)+P(1)=0$$
$$P(7)-P(6)-P(5)+P(2)=0$$

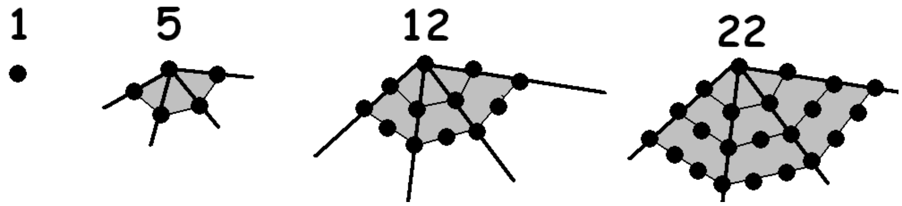
and in general:

$$\boxed{P(N)-P(N-1)-P(N-2)+P(N-5)+P(N-7)-P(N-12)-P(N-15)+\cdots=0}$$

with the convention that $P(N-k)$ is zero if $N-k$ is negative and $P(N-k)$ is 1 if $N-k$ is zero.

Although we have no explicit formula for $P(N)$ we have established that there is a curious structure among the partition numbers fundamentally connected to the numbers 1, 2, 5, 7, 12, 15, …. This wonderfully bizarre result is known as Euler's Pentagonal Number Theorem.

COMMENT: The numbers 1, 5, 12, 22, …arising as every second term in the sequence are the <u>pentagonal numbers</u>. These numbers come from arranging dots into pentagonal arrays.



The k th pentagonal is composed of a large triangle and two smaller triangles. If $T_k = 1+2+3+\cdots+k = \frac{k(k+1)}{2}$ denotes the k th triangle number, then the k th pentagonal number is $T_k + 2T_{k-1} = \frac{k(k+1)+2(k-1)k}{2} = \frac{k(3k-1)}{2}$.

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