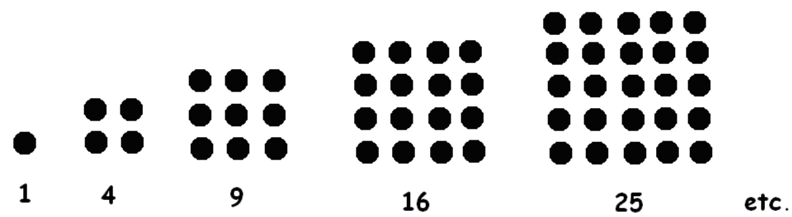
An Essay on

AVERAGE TRIANGULAR, SQUARE and PENTAGONAL NUMBERS

James Tanton
www.jamestanton.com

Here is a puzzle:

a) The square numbers 1, 4, 9, 16, 25, …. arise from arranging pebbles into squares:



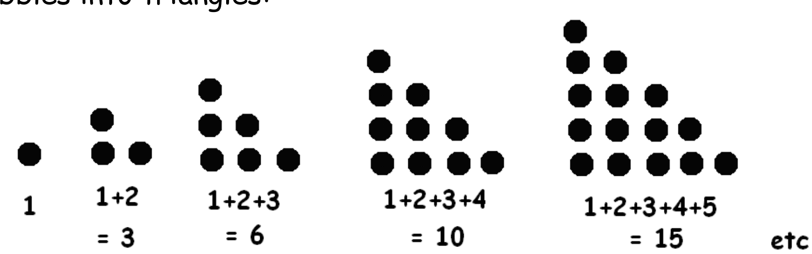
Twenty-Five (which is 5^2) happens to be the average of 1 and 49, and 100 is the average of 4 and 196.

$$5^2 = \frac{1^2 + 7^2}{2} \qquad 10^2 = \frac{2^2 + 14^2}{2}$$

$169 = 13^2$ is the average of two square numbers. Which two?
$225 = 15^2$ is the average of two squares. Which two?

Are there infinitely many square numbers that are the average of two squares?

b) The triangular numbers are the numbers 1, 3, 6, 10, 15, 21, …. They arise from arranging pebbles into triangles:



Are there triangular numbers that are average of two other triangular numbers? Infinitely many?

AVERAGE FIGURATE NUMBERS

The list of square numbers that are each the average of two squares begins

$$25, 100, 169, 225, 289, 400, 625, 676, 841, ...$$

with the ellipses at the end suggesting that this lists goes on forever. It is certainly the case that if one has one example, we can generate infinitely many more from it:

From $5^2 = \frac{1^2+7^2}{2}$ *it follows that* $(5n)^2 = \frac{(n)^2+(7n)^2}{2}$ *for each value* n.

But in some sense these new examples are uninteresting. Are there infinitely many fundamentally distinct examples? That is, can we find infinitely many solutions to

$$c^2 = \frac{d^2+e^2}{2}$$

with c, d and e sharing no common factor (other than 1)? Of the squares in the list above only four are *primitive*:

$$25 = 5^2 = \frac{1^2+7^2}{2} \qquad 169 = 13^2 = \frac{7^2+17^2}{2}$$

$$289 = 17^2 = \frac{7^2+23^2}{2} \qquad 841 = 29^2 = \frac{1^2+41^2}{2}$$

Are there infinitely many primitive examples?

SOMETHING CURIOUS: Every square that is the average of two squares is also a sum of two squares. For example:

$$25 = \frac{1^2+7^2}{2} = 3^2+4^2 \qquad 100 = \frac{2^2+14^2}{2} = 6^2+8^2$$

$$169 = \frac{7^2+17^2}{2} = 5^2+12^2 \qquad 225 = \frac{3^2+21^2}{2} = 9^2+12^2$$

$$289 = \frac{7^2+23^2}{2} = 8^2+15^2$$

It is easy to explain why this is the case.

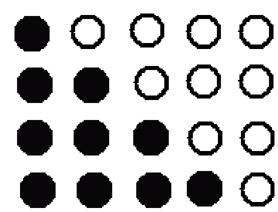
If $c^2 = \frac{d^2+e^2}{2}$*, then* d *and* e *must be both even or both odd so that* d^2+e^2 *is divisible by two. Thus* $\frac{e+d}{2}$ *and* $\frac{e-d}{2}$ *are both integers. Algebra shows* $c^2 = \left(\frac{e-d}{2}\right)^2 + \left(\frac{e+d}{2}\right)^2$ *and so we have a sum of two squares.*

The converse is also true :

If $c^2 = a^2 + b^2$*, as sum of two squares, then* $c^2 = \frac{1}{2}\left((a-b)^2 + (a+b)^2\right)$ *an average of two squares.*

So to find squares that are averages of two squares, we simply have to look for Pythagorean triples. And there are plenty of those, infinitely many primitive in fact, (see http://www.jamestanton.com/?p=628) and so there are indeed infinitely many primitive examples of average squares.

AVERAGE TRIANGLES: Placing two triangles together makes a rectangle.



2 x (1+2+3+4) = 4x5

From this we deduce that the N th triangular number, $T_N = 1+2+\cdots+N$, equals $\frac{N(N+1)}{2}$. Finding a triangular number T_c that is the average of another two, T_a and T_b requires solving:

$$2c(c+1) = a(a+1) + b(b+1)$$

This is algebraically equivalent to the equation

$$(2c+1)^2 = (a+b+1)^2 + (a-b)^2$$

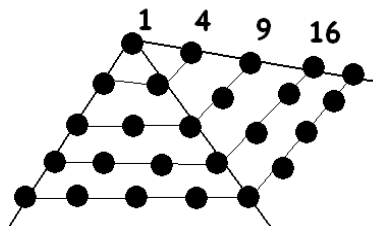
So again we are on the hunt for Pythagorean triples, but ones with odd values for the hypotenuse

.

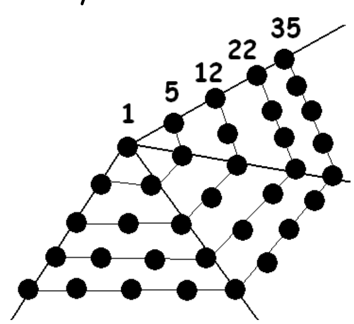
From $13^2 = 5^2 + 12^2$, for example, we obtain $c = 6,\ a = 8,\ b = 3$ and $T_6 = 21 = \frac{T_3 + T_8}{2} = \frac{6+36}{2}$. Other examples include:

$$28=\frac{1+55}{2} \qquad 36=\frac{6+66}{2} \qquad 78=\frac{3+153}{2}$$
$$153=\frac{6+300}{2} \qquad 171=\frac{66+276}{2}$$

OTHER FIGURATE NUMBERS: Squares of pebbles can be thought of as two triangular arrays glued together:



We see $N^2 = 2T_n - N$. (The shared line is double counted.) The *pentagonal numbers* P_N come from adjoining three arrays:



We have $P_N = 3T_N - 2N = \frac{1}{2}N(3N-1)$. Finding solutions to $P_c = \frac{1}{2}(P_a + P_b)$ is equivalent to solving $(6c-1)^2 = (3a+3b-1)^2 + (3a-3b)^2$ and we, again, are on the hunt for Pythagorean triples.

EXERCISE: Do find some pentagonal numbers that are the average of two pentagonals. Can you find average hexagonal numbers? Heptagonal numbers? [My thanks to Jake Wildstrom for crystallizing this idea.]

COMMENT: Some of this material appears in *THINKING MATHEMATICS: Volume I: Arithmetic = Gateway to All* available at www.jamestanton.com. This essay also appears as the February 2011 newsletter of the St. Mark's Institute of Mathematics. (www.stmarksschool.org).