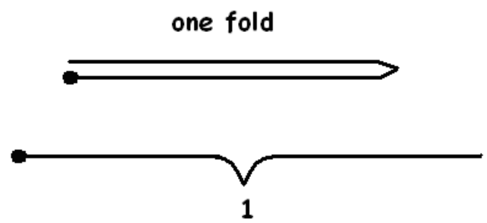
THE REMARKABLE PAPER-FOLDING SEQUENCE

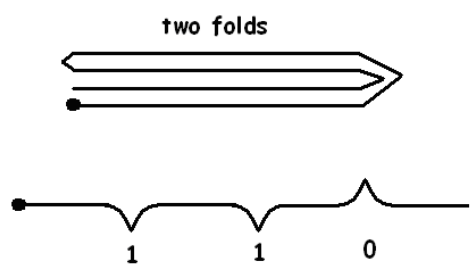
Here's a simple sequence that arises from just folding a strip of paper in half multiple times. The mathematical properties of this sequence are astounding!

THE SET-UP

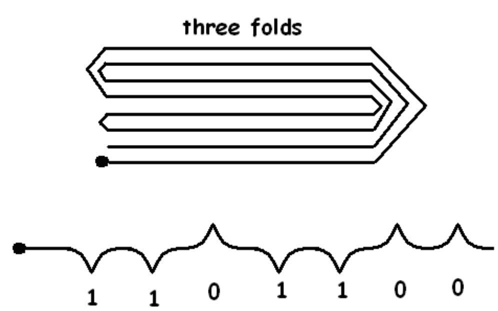
Take a strip of paper and imagine its left end is taped to the ground. If we pick up the right end, fold the strip in half, and unfold it, we find that the paper has a "valley" crease in its center. Label such a crease 1.



Now, suppose we perform two folds, always lifting the right end up over to the left end. When we unfold the paper, we find three creases: two valley creases and one mountain crease. Label the mountain crease 0.



For three folds we obtain:



The sequence for four fold turns out to be: 110110011100100.

SOME THINGS TO PONDER BEFORE READING ON

a) What is the sequence of 1s and 0s for five folds?

b) How many digits are in the one-hundred fold sequence? How many of those digits are 1s? How many are 0s?

c) Look at the sequences of numbers we have for one, two, three and four folds:

1
1 1 0
110 1 100
1101100 1 1100100

Each sequence begins with the entire previous sequence. Coincidence?

Also … The second portion of each sequence is the "dual" of the first portion: It is the first portion written backwards with the zeros changed to ones and ones changed to zeros. Is this always the case?

Comment: If this is indeed true, then all we need do to write the six-fold sequence is to write down five-fold sequence, add a 1, and then write the five-fold sequence backwards switching 1s and 0s.

d) Look at the four-fold sequence:

<u>1</u> 1 <u>0</u> 1 <u>1</u> 0 <u>0</u> 1 <u>1</u> 1 <u>0</u> 0 <u>1</u> 0 <u>0</u>

The first term is 1, the third term 0, the fifth term is 1, the seventh term is 0, and so on. The odd terms give the alternating sequence 1,0,1,0,1,0, 1,0. The even terms are 1,1,0,1,1,0,0 which is the previous fold sequence!

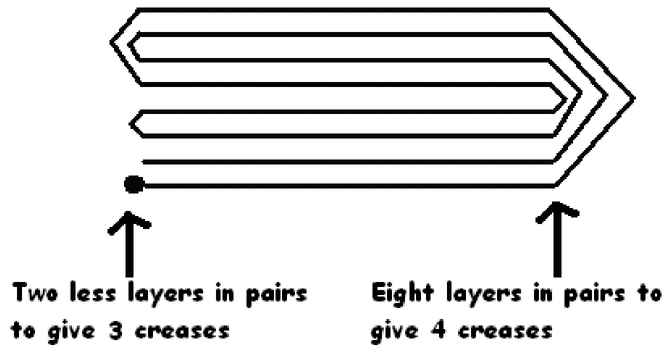
Prove that each fold sequence is the intertwining of the alternating sequence 1, 0, 1, 0, 1, 0, … with the previous sequence!

Comment: This gives another way to write the next fold sequence from a given one.

e) What is the 112^{th} digit of the one-hundred fold sequence?

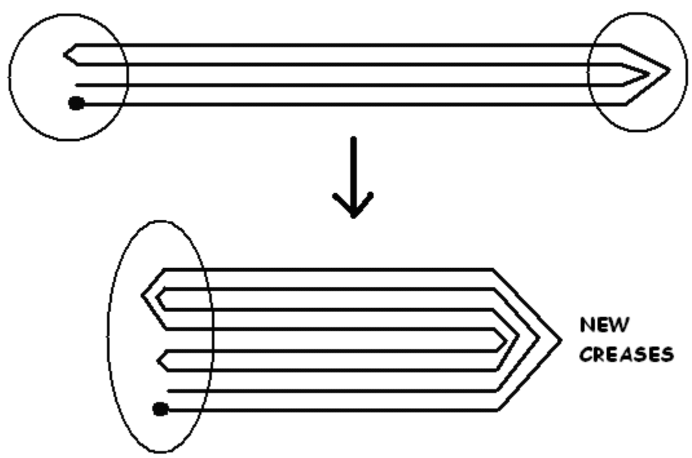
DISCUSSION AND ANSWERS:

First note, one fold produces two layers of paper, two folds four layers of paper, and in general, n folds gives 2^n layers of paper.



On the right side of the fold the layers come in pairs to make $\frac{1}{2}\cdot 2^n$ creases and on the left side there is one less crease giving $\frac{1}{2}\cdot 2^n - 1$ creases. The total number of creases after n folds is: $\frac{1}{2}\cdot 2^n + \frac{1}{2}\cdot 2^n - 1 = 2^n - 1$. Thus in the *n*th folding sequence there are a total of $2^n - 1$ zeros and ones.

The left set of creases on a given fold is just the set of creases from the previous fold. The creases on the right are the new creases.



Starting at the solid dot in the diagram, if we trace our finger along the line we first pass through a new crease on the right (upwards), then pass through the first crease of the previous fold, then through another new crease on the right (downwards), then the second crease of the previous fold, and so on. That is, we alternately pass through a new crease and an old crease. Also, we pass through the new creases alternately upwards and downwards (giving alternate valley and mountain creases) and we pass though the old creases in the same order that they were before. This explains part d).

To justify c), again imagine tracing through the line of the paper starting at the solid dot. As we move up to the middle crease, we see that we are just tracing the outline of the previous fold, and so follow the same creases as the previous fold. (Imagine that two layers of paper are glued together. This reduces the fold to the previous case.)

Adding colour makes three fold look like two folds

We then meet the middle crease, 1, and then trace the same sequence of creases in reverse order and in reverse direction (thus turning all valleys to mountains and mountains to valleys). We see:

new seqeunce = old sequence + 1 + dual of old sequence

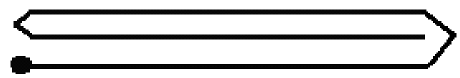
All the 1s in the old sequence are 0s in the dual, and vice versa. Thus the there are an equal number of 0s and 1s in "old sequence + dual." The 1 in the middle shows: *In any sequence there is precisely one more 1 than zeros.* Since there a $2^n - 1$ creases in all, $2^{n-1} - 1$ are zeros and 2^{n-1} are ones.

To identify particular elements in a sequence use d):

new sequence = old sequence intertwined with 1,0,1,0,...

We have: *The 1st, 3rd, 5th, 7th, etc entries of any sequence are 1, 0, 1, 0, etc. and the even entries come from the previous sequence.* In fact, entry m in the old sequence is now entry $2m$ in the new sequence. So the 112th entry in the 100-fold sequence was the 56th entry in the previous sequence, which was the 28th in the one before, which was the 14th in the one before that, which the 7th in the one before that, which was a 0. In general: *If we write $m = 2^a b$ with b odd, then the mth entry of a folding sequence is 1 if m is one more than a multiple of four and 0 if one less.*

RESEARCH QUESTION:
Care to examine the sequences that arise from repeated triple folds?



FURTHER: This paper folding exercise is intimately connected with the construction of the "Dragon Curve" fractal. Feel free to look it up on the internet.

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