EXPONENTS

Does folding a piece of paper in half multiple times explain everything there is to know about the powers of two?

TOPICS COVERED: Exponents and their properties.

A. GETTING STARTED

Here are the "doubling numbers," starting with the number 1:

1 2 4 8 16 32 64 128 256 512 1024 ...

Question 1:

a) What are the next five doubling numbers?

b) (**Tough**) Let's regard "2" as the first doubling number (it is the answer the first time "1" is doubled), "4" the second doubling number, "8" the third doubling number, and so on. Which is the first doubling number that begins with a seven? (There is one!) Which is the next doubling number that begins with a seven? Which is the third? Which is the fourth? What do you notice?

HINT: A calculator might help here! One has to go quite far along the sequence of doubling numbers to find these sevens.

Each of these doubling numbers, of course, is the result of multiplying by 2 multiple times. For example, 8 is the product of three twos:

$$8 = 2 \times 2 \times 2$$

and 1024 is the product of ten twos:

$$1024 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

It becomes tedious writing out large products like this so mathematicians have settled upon the following piece of notation:

CONVENTION: *Superscripts are used to denote repeated multiplication. For example, if a is a positive whole number, then 2^a means:*

$$2 \times 2 \times \cdots 2 \times 2 \quad (a \text{ times})$$

For example:

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$

and

$$2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$$

This convention is a confusing with regard to the number $a = 1$. Most people take it to mean:

$2^1 = 2$ (That is, one two.)

The convention says nothing to say if a is <u>not</u> a positive whole number? Is it possible to make sense of 2^0, for example? (A product of zero 2s?) Does 2^{-3} make sense? Could $2^{\frac{1}{2}}$ mean anything?

Question 2:

a) Many people say that 2^a means "multiply two by itself *a* times."

This makes sense for 2^3, for example:

$$2^3 = \text{ multiply two by itself three times } = 2 \times 2 \times 2 = 8$$

Does this way of thinking make sense for 2^1? What do you think? Seriously think about this.

Does this way of thinking make sense of 2^0, or is it of no help? How about for 2^{-1}? For $2^{\frac{1}{2}}$?

b) Some people say that a better way to think of this is to count how many times the number 1 is doubled. For example, 2^3 means "double the number 1 three times." So:

$$2^3 = 1 \times 2 \times 2 \times 2 = 8$$

Does this way of thinking give good meaning to 2^1? How about to 2^0? To 2^{-1}? To $2^{\frac{1}{2}}$?

c) Some people like to believe patterns are true and not question why they might choose to believe them. Such a person might notice:

$$\begin{aligned} &\vdots \\ &16 = 2^4 \\ &8 = 2^3 \\ &4 = 2^2 \\ &2 = 2^1 \text{ ????} \\ &1 = 2^0 \text{ ???} \\ &\frac{1}{2} = 2^{-1} \text{ ???} \\ &\frac{1}{4} = 2^{-2} \text{ ???} \\ &\vdots \end{aligned}$$

The list on the left is halving from step to step and the superscripts to the right are decreasing by one each step.

This pattern suggests it would be nice to say that $2^1 = 2$, that $2^0 = 1$, and $2^{-1} = \frac{1}{2}$. What do you think of this? Should we agree with what the pattern suggests?

Does the pattern suggest what to say about $2^{\frac{1}{2}}$?

d) Mr. Twinkletot likes to give a physical demonstration of the double numbers to his students. He takes a large piece of paper and folds it in half once. He says "Look: One fold gives two layers" and writes on the board:

$$2^1 = 2$$

Comment: Feel free to take out a piece of paper and fold it too as you read this.

He folds the paper in half a second time and says to the class "Two folds gives four layers" and writes on the board:

$$2^2 = 4$$

After a third fold he says: "Three folds gives eight layers" and writes:

$$2^3 = 8$$

Poindexter is clever and asks Mr. Twinkletot to unfold the paper and go back to the beginning. Poindexter says "Look: No folds gives one layer." Poindexter then goes up to the board and writes _________ . (Fill in the blank.)

Clarissa is even cleverer. She says that peeling the paper apart into half layers is the opposite of folding and that 2^{-1} should be the result of doing the opposite of folding one time. She writes on the board: $2^{-1} =$ __________. (Fill in the blank.)

Mr. Twinkletot is befuddled but delighted. He never thought of these ideas before.

He then asks: "Can we make sense of $2^{\frac{1}{2}}$ using this folding idea?"
What do you think? Can we?

Comment: The answer can be "no." But it would be exciting if the answer was "yes."

The numbers 1, 2, 4, 8, 16, 32, 64, … are called the *powers of two*. We feel confident to write:

$$\vdots$$
$$16 = 2^4$$
$$8 = 2^3$$
$$4 = 2^2$$

and with hesitation we might agree to write:

$$2 = 2^1$$
$$1 = 2^0$$

(but it would be nice to have a clear mathematical reason as to why we might choose to agree to this).

We should be hesitant of quantities like 2^{-1} and $2^{\frac{1}{2}}$.

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