GRAPHING COMPLEX SOLUTIONS TO QUADRATICS

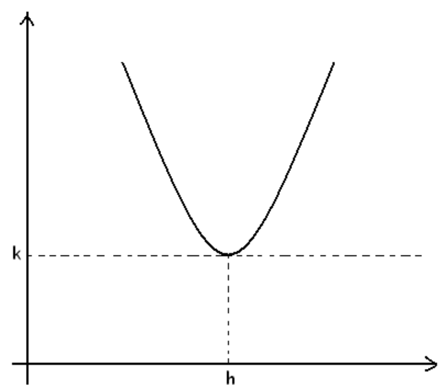
Have you ever wondered if anything can be said graphically about the complex solutions to a quadratic equation $ax^2 + bx + c = 0$ with no real solutions? Do the location of those complex roots have any connection to the graph of $y = ax^2 + bx + c$?

PREREQUISTE: This essay assumes familiarity with complex numbers. Read chapter 22 of THINKING MATHEMATICS!, Volume 2.

Any quadratic $y = ax^2 + bx + c$ can be written in the form $y = a\left(x-h\right)^2 + k$, and so is just a transformed version of the basic U-shaped curve $y = x^2$.

> [It is easy to verify the algebra here: Expand $y = a\left(x-h\right)^2 + k$ to get $y = ax^2 - 2ahx + h^2 + k$, which shows that setting $h = -\dfrac{b}{2a}$ and $k = c - \dfrac{b^2}{4a}$ does the trick!]

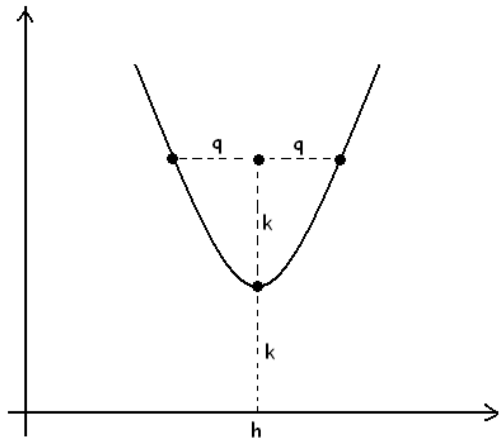
If *a* is positive, we see that the function $y = a\left(x-h\right)^2 + k$ takes all values *k* and higher, with the point (h, k) being the vertex of the parabola.



If, along with *a* being positive, *k* is positive, then graph fails to cross the *x*-axis, meaning that the equation $a\left(x-h\right)^2 + k = 0$ has no real solutions, only complex solutions:

$$a\left(x-h\right)^2+k=0$$
$$\left(x-h\right)^2=-\frac{k}{a}$$
$$x-h=\pm i\sqrt{\frac{k}{a}}$$
$$x=h\pm i\sqrt{\frac{k}{a}}$$

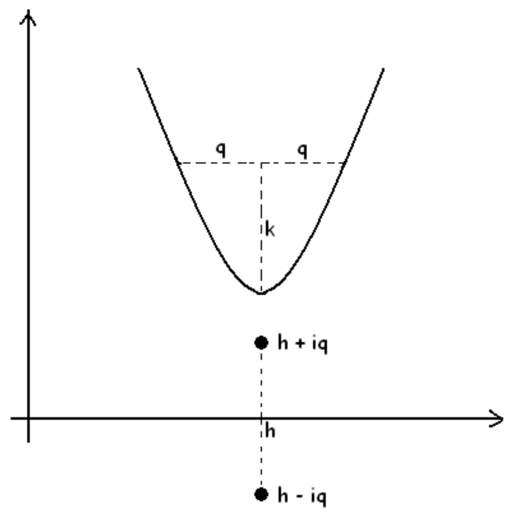
Surprisingly, there an easy way to locate those solutions on the graph!



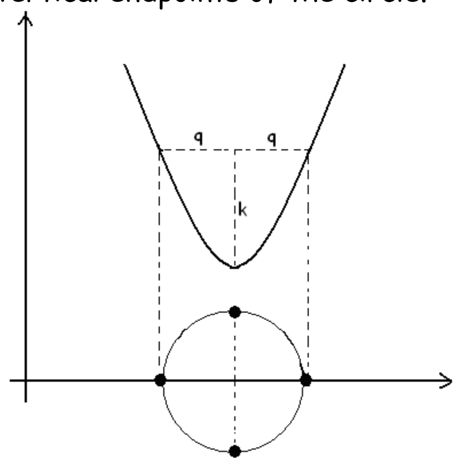
With a ruler, measure a vertical line twice the height of the vertex. Then measure the horizontal distance *q*, either left or right, to the parabola. We have $q=\sqrt{\frac{k}{a}}$.

(CHECK THIS by showing that $a\left(x-h\right)^2+k=2k$ has solutions $x=h\pm\sqrt{\frac{k}{a}}$.)

If we think of the plane of the graph as the complex plane, then the roots of the quadratic lie at the positions shown:



A swift way to locate these points is to draw a circle with its two *x*-intercepts as the endpoints of a diameter as shown. This gives a circle of radius *q* and the two complex roots lie at the vertical endpoints of the circle.



EXERCISE: Sketch the complex solutions to $y = (x-2)^2 + 4$

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FURTHER: Download the essay on "TEACHERS' – and students' – GUIDE TO EVERTHING ABOUT QUADRATICS" for further reading.

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