AN EXTRAORDINARILY QUICK WAY TO SKETCH QUADRATICS

Students in an algebra class spend many a week learning to sketch graphs of quadratic functions:

$$y = ax^2 + bx + c$$

And students are usually taught to memorise the following:

> *The graph is an upward or downward U-shaped graph (called a parabola) depending on whether the leading coefficient a is positive or negative.*
>
> *The vertex of the parabola occurs at* $x = -\frac{b}{2a}$. *Plug in this of x into the formula to find the y-coordinate of the vertex.*
>
> *Plugging in other values of x will help sketch the correct "steepness" of the U-shape. (Plugging in* $x = 0$ *is usually a good choice.)*

My first piece of advice:

FORGET THE FORMULA FOR THE VERTEX!
Memorising is joyless!

Here is an extraordinarily quick way to sketch these curves:

Prerequisite: This technique is based on the fact that that the graph of any quadratic curve $y = ax^2 + bx + c$ is guaranteed to be a symmetrical U-shape curve, upward facing if *a* is positive, downward facing if *a* is negative.
(Read the essay Why are all quadratics U-shaped? to see why this is the case.)

Let's start with an example. Consider:

$$y = x^2 + 4x + 5 .$$

We know that this is going to be an upward-facing U-shaped graph.

[Actually … if you cannot recall whether or not this will be upward facing or downward facing try plugging in $x = 1000000$. Is the answer large and positive or large and negative?]

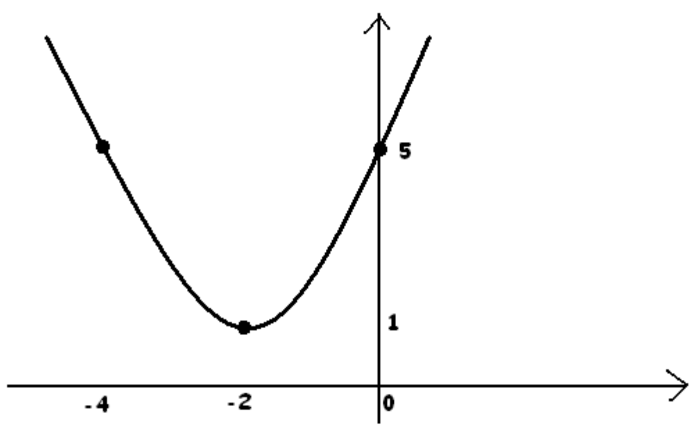
Pull out a common factor of x from the first two terms and write the expression as:

$$y = x(x+4)+5 .$$

This shows that $x = 0$ and $x = -4$ yield the same output of 5. We have two points on the parabola: $(-4,5)$ and $(0,5)$.

Since we know that the parabola is symmetrical, the vertex of the parabola must be half-way between these two x-values that yield the same output, namely, at $x = -2$. Substituting in gives the vertex at $(-2,1)$.

These three points allow us to sketch the quadratic.



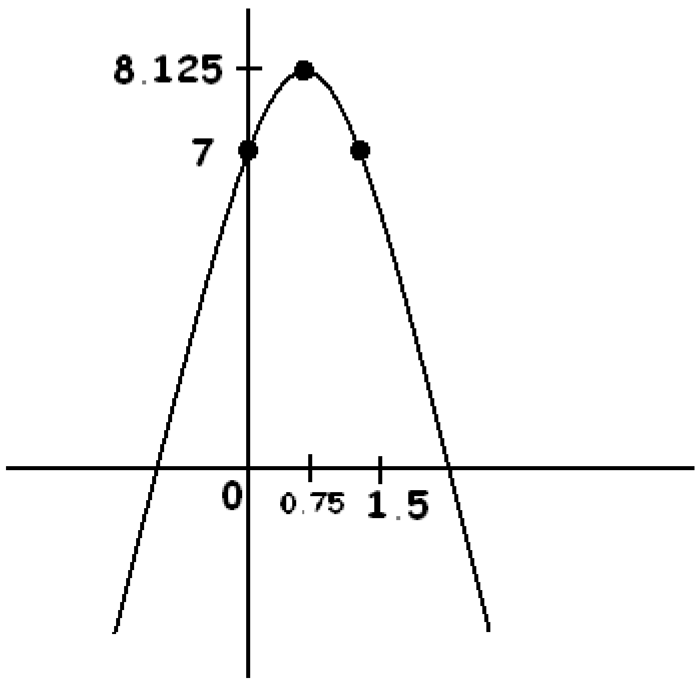
EXAMPLE: Make a quick sketch of $y = -2x^2 + 3x + 7$.

Answer: This is a downward facing parabola. We have:

$$y = -2x^2 + 3x + 7 = x(-2x+3)+7$$

and this parabola has the value 7 at both $x = 0$ and $x = \frac{3}{2}$. Because the graph is symmetrical, the vertex must be halfway between these values, at $x = \frac{3}{4}$. At this value, $y = \frac{3}{4}\left(-2 \cdot \frac{3}{4} + 3\right) + 7 = \frac{9}{8} + 7 = 8\frac{1}{8}$.

The graph appears:



QUADRATICS OF THE FORM $y = a(x-p)(x-q)$

Consider, for example, the formula:

$$y = 2(x-3)(x+8)$$

If we expand brackets we see that this can be rewritten:

$$y = 2x^2 + 10x - 48$$

and so the graph of this function is again an (upward facing) parabola.

In the same way, expanding brackets shows that $y = -3(x+4)(x-199)$ is a downward facing parabola. (CHECK THIS!)

In general:

> $y = a(x-p)(x-q)$ **is a parabola; upward facing parabola if *a* is positive, downward facing if *a* is negative.**

Quadratics that happen to be in this factored form have the nice property that one can easily read off its *x*-intercepts.

EXAMPLE: Where does $y = 2(x-3)(x+8)$ cross the *x*-axis? What is the *x*-value of its vertex? Briefly describe the graph of this function.

Answer: Can you see that $y = 0$ for $x = 3$ and for $x = -8$? Thus the graph of this function crosses the *x*-axis at these two values.

Because the graph is symmetrical (an upward facing parabola), the vertex occurs halfway between these two zeros. That is, the vertex occurs at $x = \frac{(-8)+3}{2} = -\frac{5}{2}$.
Here the *y*-value of the graph is $y = 2\left(-\frac{11}{2}\right)\left(\frac{11}{2}\right) = -\frac{121}{2} = -60\frac{1}{2}$.

Just for kicks, the *y*-intercept is (put $x = 0$): $y = 2(-3)(8) = -48$.

The graph is thus:

An upward facing parabola with vertex $\left(-2\frac{1}{2}, -60\frac{1}{2}\right)$, crossing the *x*-axis at $x = -8$ and $x = 3$. The *y*-intercept is $y = -48$

EXERCISE: Quickly sketch the following quadratics:

a) $y = 6x + x^2 - 1$
b) $y = 4x^2 + 20x + 80$
c) $y = 6 - 3x^2 - 30x$
d) $y = 2(x-5)(x-11)$
e) $y = -3(x+4)(x-4)$
f) $y = -x(x+6)$

EXERCISE: Consider the quadratic $y = ax^2 + bx + c$. Rewrite this as:

$$y = x(ax+b) + c$$

a) The *x*-coordinate of parabola's vertex lies between which two values?
b) Explain why the vertex of the parabola occurs at $x = -\frac{b}{2a}$.

COMMENT: Many teachers make their students memorise this result. For example, given $y = 3x^2 + 4x + 8$, say, they like students to be able to say that its vertex lies at $x = -\frac{b}{2a} = -\frac{4}{2\cdot 3} = -\frac{2}{3}$. If speed is important to you, then great! If not, there is nothing wrong with writing $y = x(3x+4) + 8$ and saying that the vertex is halfway between $x = 0$ and $x = -\frac{4}{3}$.

FURTHER: Download the pamphlet on **TEACHERS' – and students' – GUIDE TO EVERYTHING QUADRATIC** for further reading.

© 2009 James Tanton