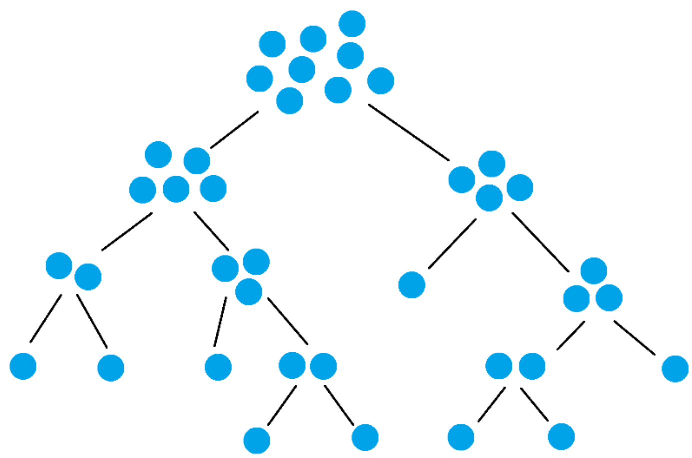
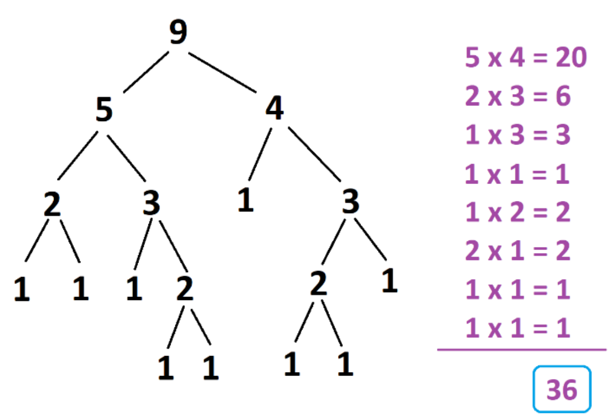
GLOBAL MATH PROJECT

PILE SPLITTING



Here's a curious puzzle. It's a classic.

> *Take a pile of nine pebbles and split them into piles. Say, piles of 4 and 5, and on the side of the page write the product* $5 \times 4 = 20$. *Now split the pile of 5 into two pile, say piles of 2 and 3 and write on the side of the page the product* $2 \times 3 = 6$. *Keeping doing this—splitting piles and writing products—until you have nine piles with 1 pebble each.*
>
> *Now sum up the products. In my example I got the magic sum of* 36.



> *Try this puzzle yourself making different choices than I did along the way. What magic sum do you get? Try it several times.*

Why do we obtain the same magic sum each and every time?
What's special about the number 36?

Some Natural Wonderings

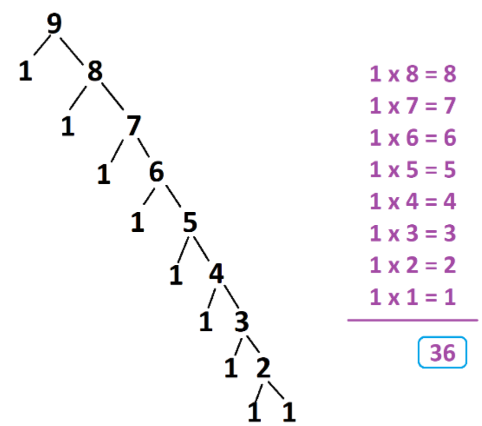
1. It is natural to want to try pile-splitting with a different starting count of pebbles. Do so!

   Show that starting with 8 pebbles always seems to lead to the same magic sum of 28.

   Show that starting with 10 pebbles always seems to lead to the same magic sum of 45.

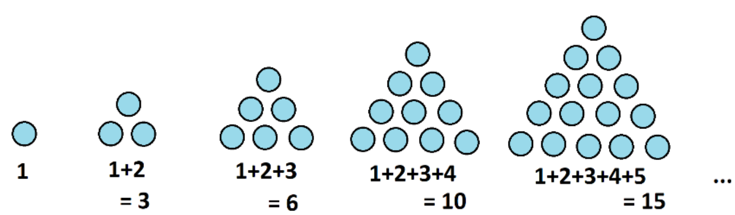
   What do the magic sums seem to be for starting with 2 through 7 pebbles? Is there structure to these numbers?

2. If you are willing to believe that pile-splitting is sure to give the same magic number each and every time, then we can detect what that number is going to be by "splitting off" only one pebble at a time.

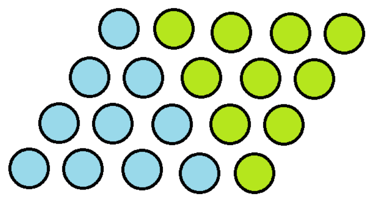


The magic number that arises starting with N pebbles is $1+2+3+\cdots+(N-1)$.

Such numbers are known as the *triangular numbers.*



The k th triangular number is half of an k-by-$k+1$ array, and so the k th triangular number if $1+2+3+\cdots+k=\frac{k(k+1)}{2}$.



A four-by-five array

Thus, ASSUMING we do indeed obtain the same magic sum each and every time, that magic sum is $1+2+\cdots+(N-1)=\frac{N(N-1)}{2}$.

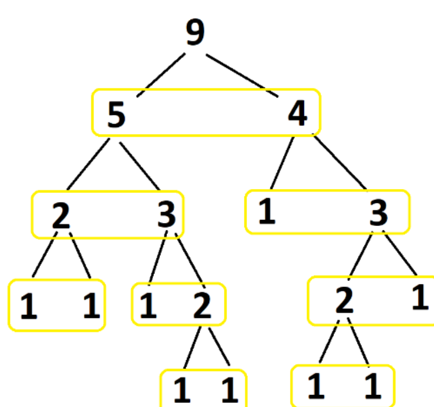
But the question remains:

Why does pile-splitting yield the same magic sum each and every time?

GOING WILD

Before working on that question, let's push this classic puzzle to new realms and ask even more questions.

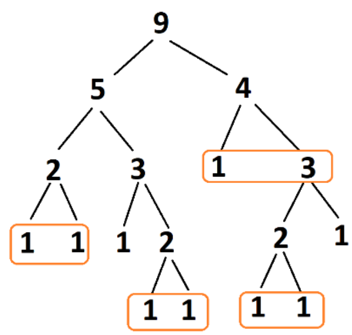
1. When I conducted my pile-splitting moves on 9 pebbles I produced 8 pairs along the way.



I bet you performed 8 splits as well.

Why, in splitting a pile of 9 pebbles, one is sure to conduct 8 splits?

2. When I conducted my pile-splitting moves on 9 pebbles I created 4 odd/odd pairs along the way.



I bet in your work you created 4 odd/odd pairs as well each and every single time.

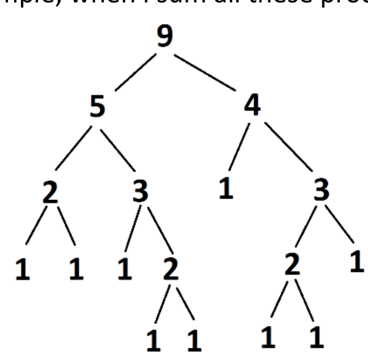
Why, in splitting a pile of 9 pebbles, the count of odd/odd pairs created along the way is sure to be 4?

How many odd/odd pairs do you invariably see in splitting a pile of N pebbles?

Let's get really crazy!

3. For a pair (b,c) that arises from splitting, instead of writing on the side of the page the product of b and c, write instead the product of a number b digits long composed solely of 2 s and a number c digits long composed solely of 2 s, namely, $\overbrace{22\cdots 2}^{b\text{ digits}} \times \overbrace{22\cdots 2}^{c\text{ digits}}$.

In my example, when I sum all these products I get 49372712. I bet you do too!

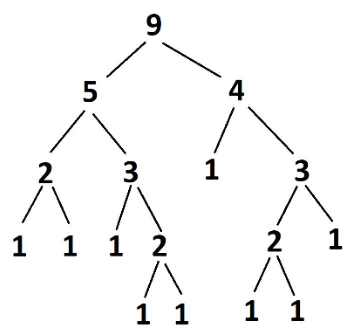


22222 x 2222 = 49377284
22 x 222 = 4884
2 x 222 = 444
2 x 2 = 4
2 x 22 = 44
22 x 2 = 44
2 x 2 = 4
2 x 2 = 4

$\boxed{49382712}$

Why is this crazy 2s version of a magic sum invariant?

4. For a pair (b,c) that arises from splitting compute the sum of fractions $\frac{1}{b}+\frac{1}{c}$. Now **multiply** together all the fractions that arise. That product is sure to be 9!



$$\frac{1}{5}+\frac{1}{4}=\boxed{\frac{9}{20}}$$

$$\frac{1}{2}+\frac{1}{3}=\boxed{\frac{5}{6}} \qquad \frac{1}{1}+\frac{1}{3}=\boxed{\frac{4}{3}}$$

$$\frac{1}{1}+\frac{1}{1}=\boxed{2} \qquad \frac{1}{1}+\frac{1}{2}=\boxed{\frac{3}{2}} \qquad \frac{1}{2}+\frac{1}{1}=\boxed{\frac{3}{2}}$$

$$\frac{1}{1}+\frac{1}{1}=\boxed{2} \qquad \frac{1}{1}+\frac{1}{1}=\boxed{2}$$

$$\frac{9}{20}\times\frac{5}{6}\times\frac{4}{3}\times 2\times\frac{3}{2}\times\frac{3}{2}\times 2\times 2=\boxed{9}$$

Why is the product of fractions created this way from pairs sure to equal the count of pebbles you start with?

And these are not the only invariants to be found in pile splitting!

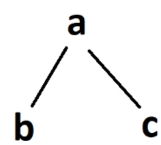
Find more!

On the next page we'll explain what is going on with these five invariants and reveal a means for you to discover many more.

PILE SPLITTING REVEALED

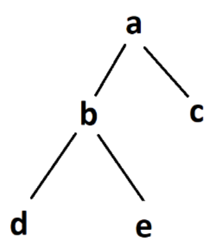
Let $A(1)$, $A(2)$, $A(3)$, … be your favorite sequence of numbers.

As you perform a pile-split



(here $b+c=a$) write on the side of the page the quantity $A(b)+A(c)-A(a)$.

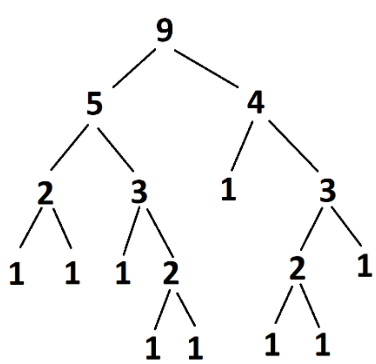
As you do this for all splits, the quantity $A(b)$ will appear with positive coefficient when it is part of the split sum, but with negative coefficient when it is a count being split.



$$A(b)+A(c)-A(a)$$

$$A(d)+A(e)-A(b)$$

Thus, when we sum all quantities of the form $A(b)+A(c)-A(b+c)$ the terms $A(b)$ corresponding to intermediary nodes of the splitting tree cancel. All that can survive are the terms $A(N)$ (with negative coefficient) and N copies of $A(1)$ with positive coefficients.



$$\begin{array}{l}
\cancel{A(5)} + \cancel{A(4)} - A(9) \\
\cancel{A(2)} + \cancel{A(3)} - \cancel{A(5)} \\
A(1) + \cancel{A(3)} - \cancel{A(4)} \\
A(1) + A(1) - \cancel{A(2)} \\
A(1) + \cancel{A(2)} - \cancel{A(3)} \\
\cancel{A(2)} + A(1) - \cancel{A(3)} \\
A(1) + A(1) - \cancel{A(2)} \\
A(1) + A(1) - \cancel{A(2)} \\
\hline
9 \cdot A(1) - A(9)
\end{array}$$

The sum of these quantities is sure to be

$$N \cdot A(1) - A(N).$$

The intermediate values that appear along the way are irrelevant.

Now it is a matter of choosing interesting sequences $A(1)$, $A(2)$, $A(3)$, …!

The Main Puzzle: Sums of Products

Let $A(n) = -\frac{1}{2}n^2$.

Then for each pair (b,c) we associate the number

$$\begin{aligned} A(b) + A(c) - A(b+c) &= -\frac{1}{2}b^2 - \frac{1}{2}c^2 + \frac{1}{2}(b+c)^2 \\ &= \frac{(b+c)^2 - b^2 - c^2}{2} \\ &= bc \end{aligned}$$

which is the product of the two numbers that appear in the pair. The sum of all these products is thus

$$\begin{aligned} N \cdot A(1) - A(N) &= -\frac{1}{2}N + \frac{1}{2}N^2 \\ &= \frac{N^2 - N}{2} = \frac{N(N-1)}{2} \end{aligned}$$

which is the $(N-1)$ th triangular number.

Question: Play with $A(n) = n^3$ or some version of it to prove that that if you associate with the pair (b,c) the product of the sum of these two numbers and the product of these two numbers (that is, associate with the pair the quantity $(b+c)\times(bc)$), the sum of all these quantities is sure to be invariant.

Counting Pairs

Let $A(n)=1$ for all n.

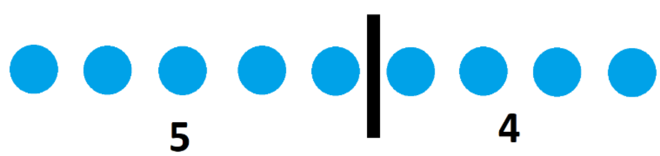
Then for each pair (b,c) the value of $A(b)+A(c)-A(b+c)$ is 1. Thus the sum of all these values counts the total number of pairs we have.

But the sum of all these values is

$$N \cdot A(1)-A(N)=N \cdot 1-1=N-1.$$

There are thus $N-1$ pairs.

Comment: There is a much more direct way to see that the number of times one performs a split on N pebbles must be $N-1$. Arrange the N pebbles in a row. Then each split corresponds to inserting a bar in a space between two pebbles.



As there are $N-1$ spaces in a row of N pebbles, there must be $N-1$ splits.

Counting odd/odd Pairs

Let $A(n)=\begin{cases}\frac{1}{2} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}$.

Then for each pair (b,c) we associate the number

$$A(b)+A(c)-A(b+c)$$

which is 0 is both b and c are even or if just one of b or c is even, but is 1 if both b and c is odd.

Thus the sum of all these values is the count of odd/odd pairs we see.

But the sum of all these values is $N\cdot A(1)-A(N)$ which equals

$$N\cdot\frac{1}{2}-0=\frac{N}{2}$$ if N is even

and

$$N\cdot\frac{1}{2}-\frac{1}{2}=\frac{N-1}{2}$$ if N is odd.

Thus the number of odd/odd pairs we see in splitting N pebbles is $\frac{N}{2}$ rounded down to an integer, if necessary.

Crazy 2s

Let $A(n) = -\frac{4}{81}(10^n - 1)$. (This strange choice will be made clear!)

Then for each pair (b,c) we associate the number

$$\begin{aligned} A(b) + A(c) - A(b+c) &= -\frac{4}{81}(10^b - 1) - \frac{4}{81}(10^c - 1) + \frac{4}{81}(10^{b+c} - 1) \\ &= \frac{4}{81}(10^{b+c} - 10^b - 10^c + 1) \end{aligned}$$

and this factors as

$$\frac{4}{81}(10^b - 1)(10^c - 1) = \frac{2}{9}(10^b - 1) \times \frac{2}{9}(10^c - 1)$$

Now

$$10^b - 1 = \overbrace{999...9}^{b \text{ digits}}$$

and so

$$\frac{2}{9}(10^b - 1) = \overbrace{222...2}^{b \text{ digits}}.$$

So here $A(b) + A(c) - A(b+c)$ matches the crazy 2 s product we described.

The sum of these products must invariably be

$$N \cdot A(1) - A(N) = N \cdot \left(-\frac{4}{81} \cdot 9\right) + \frac{4}{81}(10^n - 1) = \frac{4}{81}(10^n - 1 - 9N).$$

For $N = 9$, this has value 49372712

Challenge 1: Show that this number, in Exploding Dots notation, is $4 \mid 8 \mid 16 \mid 20 \mid 24 \mid 28 \mid 32$.

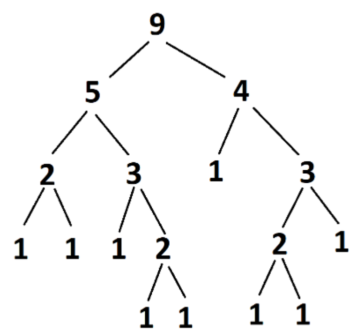
Challenge 2: Explain the Crazy 1s Puzzle in this video.

Products of Fractions

For a game with N pebbles, instead of associating with each split pair (b,c) the quantity $A(b)+A(c)-A(b+c)$ we associate the quantity

$$\frac{A(b)A(c)}{A(b+c)}$$

and take the product of all these quantities, then value of this product is sure to be $\dfrac{(A(1))^N}{A(N)}$.



$$\frac{A(5)A(4)}{A(9)} \times \frac{A(2)A(3)}{A(5)} \times \frac{A(1)A(3)}{A(4)} \times \frac{A(1)A(1)}{A(2)} \times \frac{A(1)A(2)}{A(3)} \times \frac{A(2)A(1)}{A(3)} \times \frac{A(1)A(1)}{A(2)} \times \frac{A(1)A(1)}{A(2)} = \frac{(A(1))^9}{A(9)}$$

Now choose $A(n)=\dfrac{1}{n}$.

Then we associate to each pair (b,c) the quantity

$$\frac{A(b)A(c)}{A(b+c)} = \frac{\frac{1}{bc}}{\frac{1}{b+c}} = \frac{b+c}{bc} = \frac{1}{b}+\frac{1}{c}.$$

The product of all these quantities is sure to be

$$\frac{(A(1))^N}{A(N)} = \frac{(1)^N}{\frac{1}{N}} = N.$$

Comment: This video demonstrates this Freaky Fraction Splitting.