Challenges with Multiplication

Introduction to Proofs

Overview and Outline of Lesson

Here's an age-old question: *Why is negative times negative deemed positive?*

Students in an abstract algebra class learn of the ring and field axioms and might be invited to explore why, for the ring of integers or for the field of real numbers, it must be the case that negative times negative is positive. But addressing this important question need not be relegated to a second semester advanced algebra course. It can be appropriately addressed at the beginning of an Introduction to Proofs course and so serve as a relevant introduction to systematic mathematical thinking. (But, of course, this lesson can be used as part of an Abstract Algebra course too.)

1. **Pre-Activity Assignment**
   Undergraduates complete this assignment prior to lesson. This activity explores several models for the multiplication of positive integers that call into question the idea that multiplication should be commutative. The cognitive disequilibrium induced here is deliberate.

2. **Class Activity**
   This activity illustrates the action of identifying core beliefs (axioms) of a mathematical system and proving logical consequences of these axioms. Three core beliefs that lead to the product of two negative numbers being positive are identified.

3. **Homework**
   The work here reinforces the principles of the lesson.

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College Curriculum

The work of this lesson naturally fits into an Introduction to Proofs course. It asks undergraduates to identify core beliefs (axioms) of a mathematical system and explore their logical consequences. This lesson also motivates the thinking and work of an abstract algebra course.

Connections to School Mathematics

The universal "inner workings" of arithmetic in school mathematics are often overlooked. Students are usually presented with different models to motivate various arithmetical operations—the product of negative numbers, the distributive property of multiplication over addition, for instance—in an ad hoc manner and the idea that there are fundamental common structures to these models is buried. This lesson prompts the meta-cognitive purpose of stepping back from these models to see K-12 arithmetic from an advanced perspective.

Lesson Preparation

Prerequisite Knowledge

Nominal

Learning Objectives

Undergraduates will be able to:

- Explore the logical consequences of a finite set of axioms.

Anticipated Length

One 60- to 70--minute class period.

Materials

1. Pre-Activity Assignment (to be assigned as homework for the class prior to class activity)
2. Class Activity Handout
3. Exit Ticket
4. Homework Questions

Annotated Lesson Plan

In the lesson plan below, we use the following formatting conventions:

- Regular font to indicate actions we recommend you take.
- Italicized font to indicate important points for you to consider.
- Boxes to contain the activity/ homework problem alluded to the other annotations

Professor Teaching Notes and Annotations

Before the Lesson

Give undergraduates the Pre-Activity Assignment to complete as homework in preparation for the lesson. See Solutions - Pre-Activity Assignment for solutions.

> *We recommend that you collect this pre-activity assignment the day before the lesson so that you can review students' responses before you start the class activity. This will help you assess students' writing styles and effectiveness in communicating mathematics, and will also allow you to notice innovative approaches that might be worth sharing with the class as a whole.*

Review Pre-Activity Assignment: (~10-15 minutes)

Go over the solutions to the pre-activity assignment as needed. In particular, invite students to share their thoughts, reactions, and processes for analyzing select problems.

The solution set can be shared with students, but you might wish to annotate it and share some alternative approaches students took to certain questions.

The goal of the pre-activity is to help students "step back" from procedural mathematics thinking and doing, and to begin to probe deep questions about mathematical structure. This work might be unfamiliar and a little disquieting to some.

Perhaps mention that a key goal of our work is to also practice producing effective pieces of communication that convince your peers (folk of similar mathematical background and possessing similar background knowledge) that what you claim to be so actually is so. Moreover, such communication should be "kind" to the reader, not onerous to process and, in fact, a delight to read.

Some details:

> **Question 1**: This question provides an opportunity to discuss what "explain" means. This question is more of a philosophical question and it could be daunting for students. "What am I actually expected to write?"
>
> **Question 2**: Acknowledge the surprise here that the traditional algorithm for long multiplication does not, in and of itself, imply that multiplication should be commutative.
>
> **Question 3**: One might argue that a solid mathematical proof here requires an argument by induction. However, the goal here is challenge students to write a convincing piece of communication. Identifying a pattern and making it clear that "and so on" is indeed reasonable to say (the pre-cursor to an induction argument) is likely sufficient communication here. Of course, students might present alternative arguments to the one presented in the solution set.
>
> **Question 4**: Noticing the symmetry of the situation in part a) helps students "step back" form the immediacy of the mathematics. Analysing part b) will likely feel familiar territory for students.

Motivate the Class Activity:

- Ask your undergraduates: *Have you ever wondered why negative times negative is positive? Could you explain to a friend why this must be so?*

> **<u>An Important Connection for Teaching to Make</u>:**
> It can often appear that mathematics is a collection of immutable "facts" that cannot be changed or adjusted. But an important goal in mathematics is to identify the core "axioms" for a given mathematical system (the arithmetic of numbers, the geometry of shape and symmetry, for instance) that seem to make that system work the way it does – and then to examine the possibility of finding variations of those systems that satisfy alternative, but

analogous, axioms. Identifying core axioms relies on answering the question: What properties seem to be at play in the system and what motivates me to home in those particular properties? Such meta-analysis brings to light the reasons certain choices are made in presenting the material of high-school mathematics.

Class Activity (~45 minutes)

Pass out Class Activity. See Class Activity Solutions for sample solutions.

There are a total of 10 questions in this activity. The goal is to have students work through and discuss in groups the first 8 questions. Make this clear to that students that this is the goal. (The final two questions may be considered optional, or to be completed outside of class.)

Each question is "small" in content but deep in the philosophical importance.

As groups discuss the questions, roam the room and listen in to the various conversations. Chime in with questions for each group or ask for clarifications on what you hear being discussed.

Questions 1 through 4
Once you feel all groups have attended to the first four questions, perhaps bring the class's attention back together as a whole and recap some of the ideas and thoughts the groups discussed.

Questions 5 and 6
Once you feel all groups have had the chance to examine these two questions, bring the class' attention back together again. Question 6 is looking for a formal axiomatic proof of sorts. Ask some students to present their argument for the whole class or present a proof yourself as prompted by the student thinking you overheard.

Questions 7 and 8
Question 7 gives a convincing, informal argument as to why negative times negative is positive. Perhaps ask students to analyze where the formal axioms lay behind-the-scenes in this informal approach. Question 8 then leads students through the formal work. (As a **bonus question** you can ask students to develop a formal argument explaining why -0 must be 0.)

Question 9 and 10
These optional questions push the area model to work with polynomials and infinite series. As the previous eight questions establish that area-model diagrams represent arithmetic truth (despite showing negative side lengths and negative areas) we can use these diagrams to conduct valid arithmetic and algebraic work. These two questions show the power of that work.

<u>An Important Connection for Teaching to Make</u>:

So called "elementary" questions – such as why is negative times negative positive? – require sophisticated, subtle analysis. No one model of an arithmetical system explains all aspects we like to believe about the mathematics of arithmetic and, in the end, explanations of fundamental queries must be based on logical consequences of the core axioms we choose to believe. School teachers are then faced with the difficult challenge of which tack to take in attending to such questions from students: develop a formal axiomatic approach, develop a convincing informal argument with the correct fundamental axioms hidden behind-the-scenes, or to simply point out the pattern or structure one model seems to imply. In different contexts and with different audiences, each of the three approaches might have a valid role.

Wrap-Up & Exit Ticket (~5 minutes)

For the final few minutes of session, bring the class's attention back together and ask folk to share or summarize what they felt they learned during this session.

Hand out the exit ticket and have students respond to the two prompts there. Collect the tickets and perhaps report at the start of the next session any interesting themes or ideas that emerged from the responses.

Homework Questions

At the end of the lesson, assign undergraduates all of the problems from the Homework Handout. Collect their responses. See homework solutions for sample solutions.

Question 1: This question reinforces the subtlety, and difficulty, of reconciling different models for particular arithmetic thinking.

Question 2: This question asks students to examine another unusual means to compute long multiplication and explain why the method works.

Question 3: This question brings as back to a pre-assignment question and asks students to explore its general robustness.

Question 4: This question returns a question left hanging from the class-activity: Are additive inverses unique? This provides another opportunity for students to attempt a formal axiomatic-type argument.

Assessment Questions

By the end of the unit, feel free to include the following two Assessment Questions in a quiz or exam. See Assessment Solutions for sample solutions. Collect undergraduates' responses.

Pre-Activity Assignment

Young children learn to count discrete objects and so first encounter the numbers 1, 2, 3, 4, …, the "counting numbers," as objects of mathematical study and play.

The addition of two counting numbers manifests itself as a physical action: to compute $2+3$ combine a group of two blocks with a group of three blocks to count a total of five blocks.

> **Question 1:** *Explain why the following picture demonstrates that* $2+3$ *and* $3+2$ *must, philosophically, have the same answer.*
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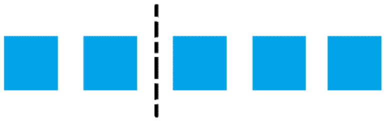
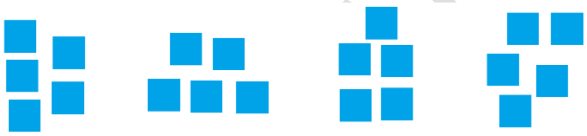
> 
>
> *Further, go on to explain why it seems reasonable to believe that* $a+b$ *should equal* $b+a$ *for all counting numbers* a *and* b.

In this world of counting numbers, multiplication manifests itself as "repeated addition." For instance, 4×5 is interpreted as "four groups of five." This gives twenty objects.



But this definition of multiplication is inherently asymmetrical: 5×4 is "five groups of four," which seems fundamentally different from "four groups of five." To make this point more dramatic, is it immediately obvious that "113 groups of 273" objects should give the same total count of objects as "273 groups of 113"?

> **Question 2:** *Use the traditional long multiplication algorithm to compute* 273×113 *and then use it again to compute* 113×273. *Show your work. Both computations give the same final answer (it's* 30849*), but does it seem obvious to you that they would?* (This is a subjective question. Please just share your thoughts and honest reactions here.)
>
> **Question 3:** *Here is an unusual geometric way to compute the product* $a\times b$ *of two counting numbers* a *and* b. *First draw* a *concentric circles. Then draw* b *radii from their common enter to the outermost circle. Then count the number of finite regions that result. That count is* $a\times b$.
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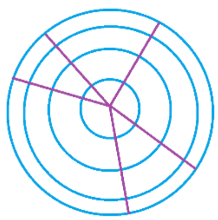
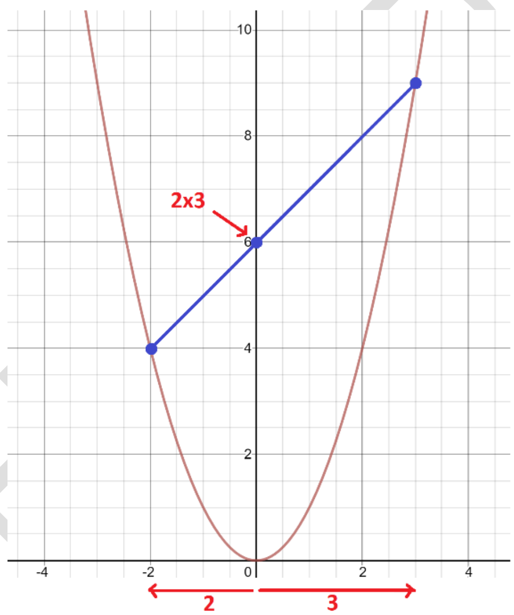
> 
>
> "4x5" gives twenty regions
>
> a) *Does it seem obvious to you that a picture of* a *concentric circles and* b *radii should yield the same count of regions as a picture of* b *concentric circles and* a *radii?* (Again, another subjective question.)

b) *Is it true that a picture of a concentric circles and b radii is sure to have $a \times b$ regions? If so, convince me!*

Question 4: *Here's another geometrical way to compute the product $a \times b$ of two counting numbers a and b. It is clearly symmetrical and so it is clear that $a \times b$ is sure to equal $b \times a$ in this approach.*

Start by drawing a symmetrical graph of the equation $y = x^2$. To compute $a \times b$ march a units to the left on the x-axis and locate the point $\left(-a, a^2\right)$ on the graph and march b units to the right on the x-axis and locate the point $\left(b, b^2\right)$ on the graph. Next, connect these two points with a line segment. Determine where this line segment intercepts the y-axis. This y-intercept is sure to be $a \times b$.

This picture shows the line segment that results for $a = 2$ and $b = 3$.



a) *Do you believe that the line segment connecting $\left(-a, a^2\right)$ and $\left(b, b^2\right)$, used for computing $a \times b$, is sure to have the same y-intercept as the line segment connecting $\left(-b, b^2\right)$ and $\left(a, a^2\right)$, used for computing $b \times a$?* (This is a yes/no question. Answer it honestly.)

b) *Explain why the y-intercept of the line segment connecting $\left(-a, a^2\right)$ and $\left(b, b^2\right)$ is sure to be $a \times b$.*

In our next class we'll examine what it takes to extend the notion of multiplication to numbers beyond counting numbers to include multiplication of all positive and negative integers. (It is not immediately clear if any of the models for multiplication presented above naturally extend.) We all "know" that negative times positive is positive, positive times negative is negative, and that negative times negative is positive, but what is it in the mathematics of multiplication that forces us to conclude this?

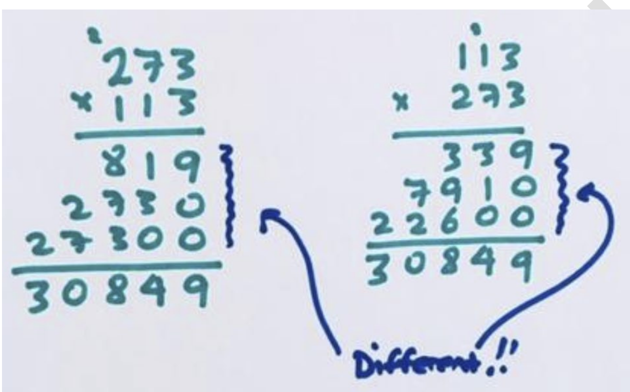
Let's find out!

PRE-ACTIVITY SOLUTIONS:

Question 1:
Looking at the picture from left to right we see two blocks followed by three blocks to represent the sum $2+3$. Looking from right to left we see three blocks followed by two blocks to represent $3+2$. Either way, it is the same picture and the total count of blocks is fixed. So it must be the case that $2+3$ and $3+2$ represent the same total.

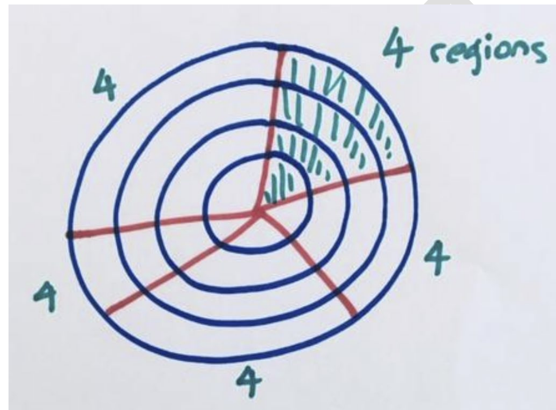
The numbers 2 and 3 are immaterial here. We see in our minds' eye an analogous picture, and follow-on reasoning, applies to any pair of counting numbers a and b.

Question 2: The two computations do look different in their execution!

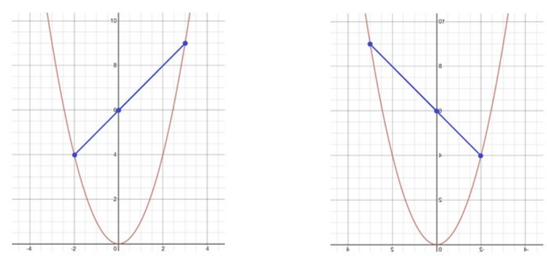


Question 3: a) It does not personally seem obvious to me that switching the counts of circles and radii does not change the count of regions!

b) With a circles each sector between two radii contains a regions. If there are b radii, then there are b sectors, giving a total of $b \times a$ regions.



Question 4: a) The graph of $y = x^2$ is symmetrical about its vertex. If we flip the image that computes $a \times b$ we obtain the image that computes $b \times a$. The y-intercept does not change, and so both computations have the same y-intercept.

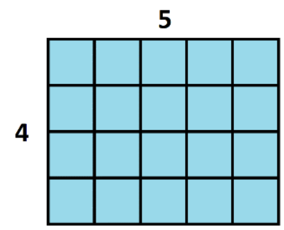


b) A line connecting $\left(-a, a^2\right)$ and $\left(b, b^2\right)$ has slope

$\dfrac{b^2 - a^2}{b-(-a)} = \dfrac{(b-a)(b+a)}{b+a} = b - a$, and so has an equation of the form $y = (b-a)x + k$ for some value k, which is the value of the y-intercept. Substituting in $x = b$, $y = b^2$ gives $k = ab$.

Class Activity

In the world of positive integers (that is, the "counting numbers," 1, 2, 3, 4, …) multiplication can be modeled via geometric area. If one unit square is considered the fundamental object, then 4×5 is represented as the count unit squares that sit inside a four-by-five rectangle, that is, 4×5 is the area of that rectangle.



This matches the idea of modeling multiplication of counting numbers as "repeated addition" for the very young: look across the rows of the rectangle and see four groups of five unit squares.

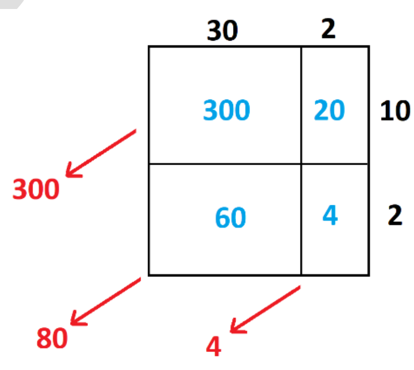
Question 1: *With the area model for multiplication, why does it seem reasonable to believe that* $a \times b = b \times a$ *for all counting numbers* a *and* b*?*

Question 2:
a) Does the area model suggest a possible meaningful value for $a \times 0$ *for a fixed counting number* a*? Does this match the thinking "repeated addition" would suggest?*

b) Kiran is thinking in terms of "repeated addition" and suggests that $0 \times 3 = 2$*. To convince you he simply places two pencils in front of him and asks: How many groups of three pencils do you see?*

What do you think of Kiran's argument?

The area model of multiplication for positive integers provides the means to compute large with some ease. For instance, to compute 32×12 we can divide a 32-by-12 rectangle into four convenient pieces.



Question 3:
a) *Compute* 32×12 *and* 12×32 *using the traditional algorithm each time. Compare your work with the area model computation.*
b) *Use the area model to compute* 273×113*, dividing a rectangle into nine convenient pieces. Compare your work with the work of the traditional algorithm.*

Question 4:

a) *The "distributive rule" states that* $a(b+c)=ab+ac$. *Show how the area model suggests this as valid, at least for counting numbers* a, b, *and* c.

b) *Use an area-model picture to expand the product* $(a+b+2)(a+c)$.

c) *The product* $(8+1+1)(3+7)(4+1+5)$ *is just* $10\times10\times10=1000$. *But is there an area-model picture (or a picture for some natural extension of the are model) for this three-fold product that computes the answer* 1000 *as a sum of small products?*

Declaring What We Choose to Believe

The area model suggests three fundamental properties of the arithmetic of counting numbers. For all counting numbers a, b and c we have

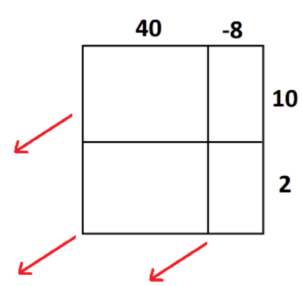
i) $ab=ba$
ii) $a\times0=0$
iii) $a(b+c)=ab+ac$

But, of course, there are other types of numbers than just the positive integers. One can conceive of rectangles with fractional side-lengths and fractional areas, and the area model seems to suggest that our three arithmetic properties should hold for positive fractions as well.

And in our mind's eye we can conceive of rectangles with irrational side lengths, and it seems that the area model suggests our three arithmetic properties should, in fact, hold for all positive real numbers.

These three arithmetic properties come to feel so "natural" and "right" that it seems compelling to believe that they should hold for ALL types of numbers, even for numbers which cannot be the geometric side lengths of rectangles, such as negative numbers and complex numbers.

Question 5: a) *Use the area model to compute* 32×12 *again, but this time think of* 32 *as* $40+(-8)$ *leaving* 12 *still as* $10+2$. *Does the answer* 384 *still result?*



Of course, the previous question assumes that we know positive times negative to be negative, as is negative times positive. How do we know this?

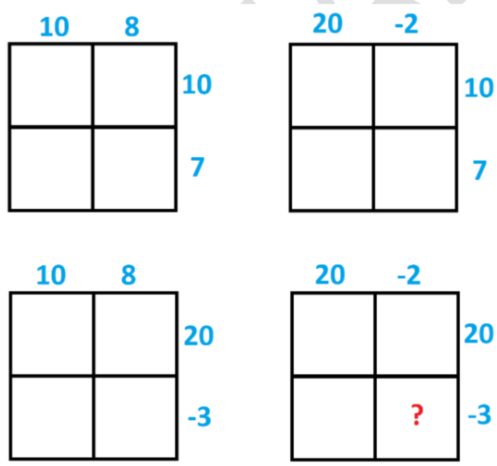
Question 6: *Let's do assume that the three arithmetic properties listed above hold for all numbers. And just to be very clear, for a counting number* a *we mean by "*$-a$*" a number with the property that* $a+(-a)=0$.

a) *Use the three properties to compute* $2\times 3+2\times(-3)$, *explicitly stating which property is being used when as you go along. Deduce that* $2\times(-3)$ *is indeed* <u>*negative*</u> *six.*
b) *Show how to deduce that* $(-2)\times 3$ *is* <u>*negative*</u> *six using the arithmetic properties.*
c) *A young student might argue that* $2\times(-3)$ *being negative follows from "repeated addition" thinking. Does this mode of thinking explain why* $(-2)\times 3$ *should be negative too? Discuss.*

Now to attend to the age-old question: Why is negative times negative positive?

Question 7: *A young student is comfortable with the ideas that positive times positive is positive, positive times negative is negative, and negative times positive is negative (via question 6c) but wants to know why negative times negative should be positive. Why is* $(-2)\times(-3)$ *positive six, for instance?*

Rakhi suggests the student try computing 18×17 *four different ways using the area model. How exactly does this exercise suggest that mathematics wants* $(-2)\times(-3)$ *to be* $+6$*?*



Question 8: *Let's make explicit the fundamental beliefs of arithmetic that lie behind-the-scenes in the previous exercise.*

a) *Examine* $(-2)\times(3+(-3))$ *using the three arithmetic properties to deduce that* $(-2)\times(-3)$ *must be* <u>*positive*</u> *six.*

Let's now be more formal and abstract.

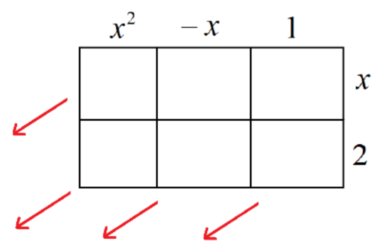
b) *If* a *is a real number, explain why* $-(-a)$ *is* a. *(HINT: What number adds to* $-a$ *to give zero?)*
c) *If* a *and* b *are real numbers, explain why*
   I: $(a)\times(-b)=-(ab)$
   II: $(-a)\times(b)=-(ab)$

III: $(-a)\times(-b)=ab$.

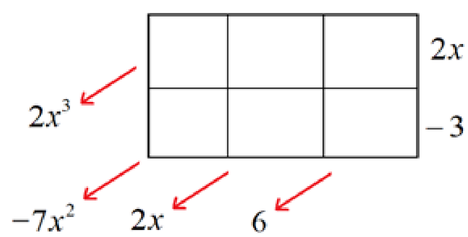
The previous questions establish that area-model diagrams represent arithmetic truth, despite possibly showing negative side lengths and negative areas. Consequently, we are permitted to use such diagrams to conduct valid work in algebra too without concern as to the sign of the numbers represented by the variables.

Question 9:

a) *Use the area model to compute* $(x^2-x+1)(x+2)$.



b) *Use the area model to compute* $\dfrac{2x^3-7x^2+2x-6}{2x-3}$.

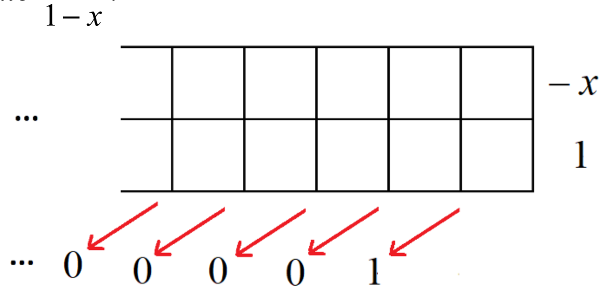


c) *Use the area model to compute* $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$.

d) *Use the area model to compute* $\dfrac{x^8-1}{x+1}$. *Also, find a factor of the number* 3036^8-1 *that is four digits long.*

Question 10:

a) *Use the area model to compute* $\dfrac{1}{1-x}$.



b) *Use the area model to compute* $\frac{1}{1-x-x^2}$.

CLASS-ACTIVITY SOLUTIONS:

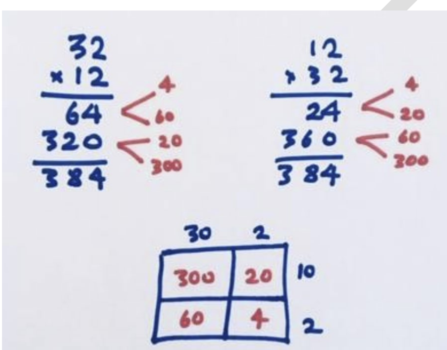
Question 1:
Rotating a rectangle 90 degrees converts an a-by-b rectangle into a b-by-a rectangle. As area does not change under rotation, we have that $ab = ba$ in this model of multiplication.

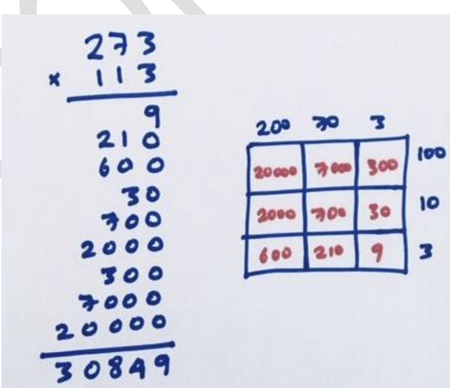
Question 2: a) Perhaps. A “rectangle” of width zero has zero area. (Though some might object to calling such a degenerate object a “rectangle.”)

If a is a counting number, then “a groups of 0” is $0+0+\cdots+0$, which is zero.

b) This is a place where the “repeated addition” model for multiplication starts to become troublesome. Among two pencils, it is true one does not see a group of three. (Though, might you argue that there is a fractional group of three, if you allow fractions in your thinking?)

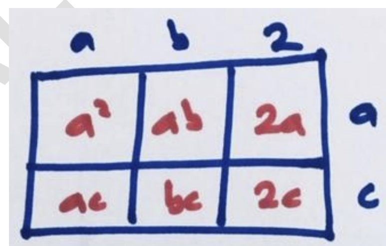
Question 3: We see that the traditional algorithm is just a compact coding of the area model approach.





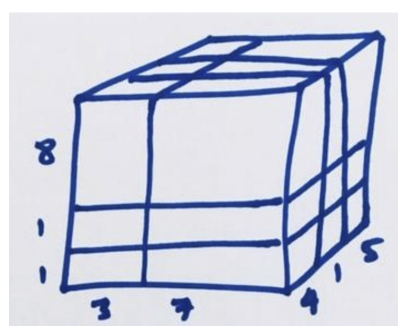
Question 4: a) The following picture demonstrates the distributive rule.

b) We get $a^2 + ab + ac + bc + 2a + 2c$.

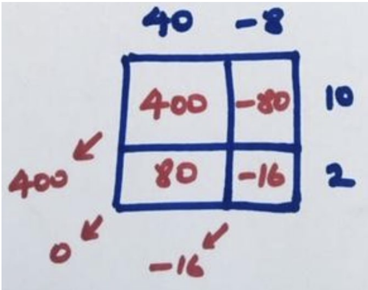


c) Divide a cube into 18 pieces as shown. Then

$$(8+1+1)(3+7)(4+1+5)$$
$$= 8\times3\times4 + 8\times7\times4 + 1\times7\times5 + \cdots$$



Question 5: It does! Despite the picture not being valid geometrically, it is still depicting arithmetic truth.



Question 6:

a) We have

$2\times 3+2\times(-3)=2\times(3+(-3))$ property iii)

$=2\times 0$

$=0$ property ii)

Thus $2\times(-3)$ is a number which, when added to $2\times 3=6$, gives zero. It must be the case then that $2\times(-3)$ is -6.

Challenge: *Could there be two different numbers that a and b with the property $6+a=0$ and $6+b=0$? How do we know that there is only possible number that deserves to be called "-6"? Let's explore this subtle issue in depth in the homework exercises.*

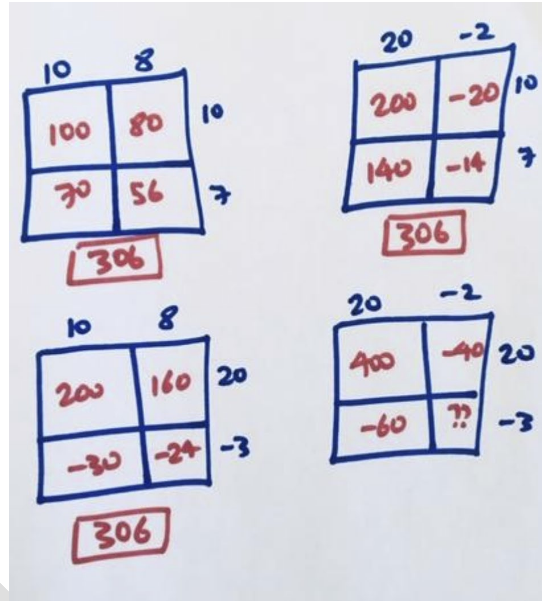
b) We have $(-2)\times 3=3\times(-2)$ by property i). Mimicking the work of part a) we see that $3\times(-2)$ is a number, which when added to 6, gives zero. It must be -6.

c) Perhaps one can argue that $2\times(-3)$ is "two groups of negative three" and so equals $(-3)+(-3)=-6$.

It is not clear what "negative two groups of three" means. But if we believe that multiplication is commutative, then we can think of $(-2)\times 3$ as $3\times(-2)=(-2)+(-2)+(-2)=-6$.

Question 7:

Mathematics clearly "wants" 18×17 to be 306.



The only way this can be in the fourth diagram is for $(-2)\times(-3)$ to be $+6$.

Question 8:

a) We have $(-2)\times(3+(-3))=(-2)\times 0=0$ by property ii).

On the other hand

$(-2)\times(3+(-3))=(-2)\times 3+(-2)\times(-3)$

by property iii). And we know that $(-2)\times 3=-6$ from the previous question. So this reads

$(-2)\times(3+(-3))=(-6)+(-2)\times(-3)$.

Putting these together gives $0=(-6)+(-2)\times(-3)$.

Thus $(-2)\times(-3)$ is a number, which when added to -6, gives zero. The number must be $+6$.

b) By definition, $-a$ is a number which, when added to a, gives zero.

$$a+(-a)=0.$$

Also, $-(-a)$ is a number, which when added to $-a$, gives zero. From the previous equation we see that a is such a number. So $-(-a)$ is a.

Comment: Again we are assuming that for each real number a there is only one possible number deserving to be called "$-a$." This issue will be explored in the homework exercises.

c) I: Using properties iii) and ii) we have

$$\begin{aligned} a\times(-b)+a\times b &= a\times\left((-b)+b\right) \\ &= a\times 0 \\ &= 0 \end{aligned}$$

showing that $a\times(-b)$ is a quantity, which, when added to ab, gives zero. It thus matches $-(ab)$.

II: Using property i) and the result we just proved we have

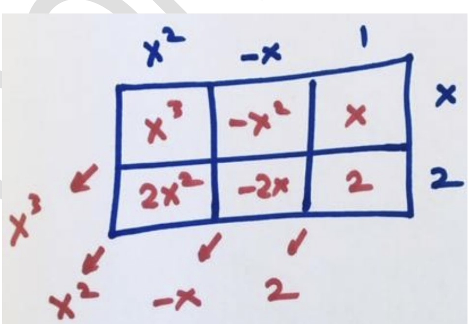
$$\begin{aligned} (-a)\times b &= b\times(-a) \\ &= -(ba) \\ &= -(ab) \end{aligned}$$

III: Using results I and II we have

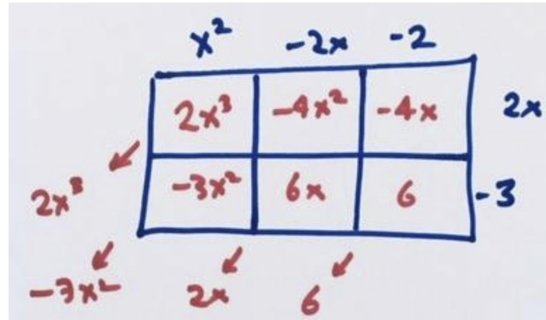
$$\begin{aligned} (-a)\times(-b) &= -\left(a\times(-b)\right) \\ &= -\left(-(ab)\right) \end{aligned}$$

and by part b) this equals ab.

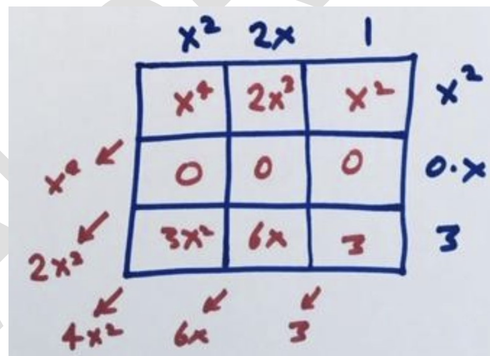
Question 9: a) We get x^3+x^2-x+2.



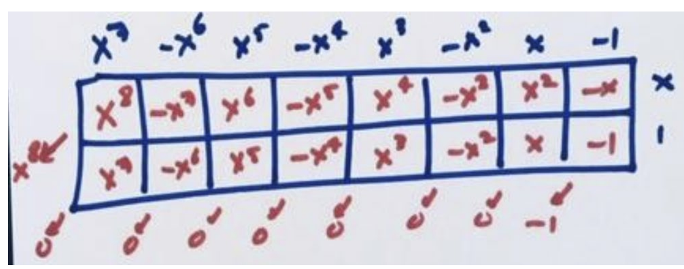
b) Starting with the top left corner we can deduce the entries of the table. We see the answer x^2-2x-2 appear.



c) Can you reason that we'll need a table with three columns?

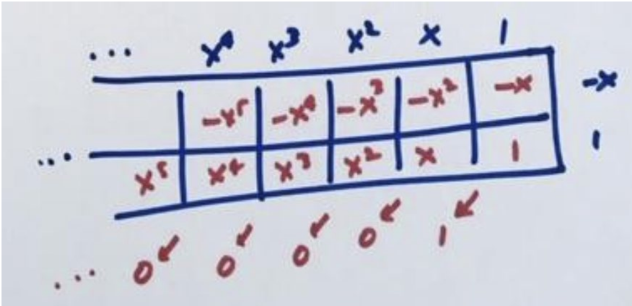


d) We see $x^8-1=(x+1)\left(x^7-x^6+x^5-x^4+x^3-x^2+x-1\right)$.



Thus $3036^8-1=(3036+1)(\textit{something})$. We see the four-digit factor 3037.

Question 10:
Start at the bottom right corner this time. We get $\dfrac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots$, the famous geometric series formula.



b)

$$\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \cdots$$

We see the famous Fibonacci numbers appear as coefficients.

Exit Ticket

1. Were you surprised by anything in today's lesson?

2. How might you explain to a young student why mathematics wants $(-4)\times(-5)$ to be $+20$?

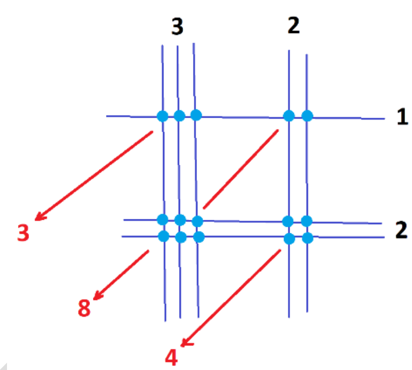
Homework Questions

Question 1: There are two standard ways to interpret division of counting numbers. For example, in a "sharing model," $20 \div 5$ is the count of objects each student receives if twenty objects are equally distributed among five students. In a "grouping model," $20 \div 5$ is the number of groups of five objects one can find in a set of twenty objects. It is surprising that both modes of thinking yield the same value 4.

If a and b are counting numbers with a a multiple of b, how might you explain why both modes of thinking are sure to yield the same numerical value for $a \div b$?

Question 2: Here's an unusual way to perform long multiplication.

To compute 32×12, say, draw sets of vertical lines matching the digits of the first number of the product and sets of horizontal lines matching the digits of the second number of the product. Add the counts of intersection points in each cluster, diagonally, and read off the answer 384!

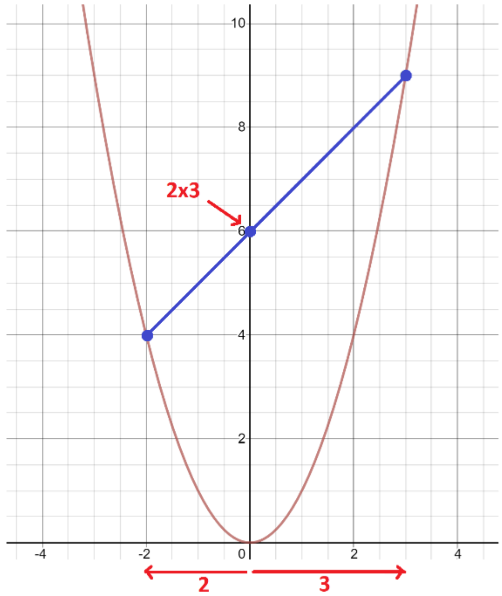


a) Use this curious method to compute each of the following products. (Check your answers with a calculator if you like.)

   i) 312×21
   ii) 243×31
   iii) 302×210

b) Explain why this curious method works.

Question 3: Recall the "parabola" method of multiplication from the pre-activity.

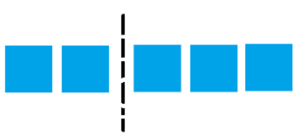


Explain how one computes $(-2)\times 3$ and $2\times(-3)$ with this model.

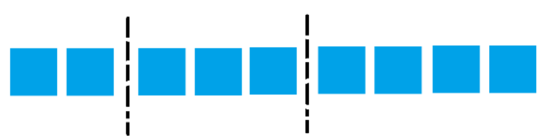
What value for $(-2)\times(-3)$ does this model give?

Question 4:

a) By looking at the picture on the left from left to right and then from right to left, we are led to believe the commutativity rule, $a+b=b+a$, for the addition of counting numbers. Explain how the picture on the right suggests an associative rule, $(a+b)+c=a+(b+c)$, for the addition of counting numbers.



$2+3=3+2$



$(2+3)+4=2+(3+4)$

If we choose to believe that the commutativity and associativity rules for addition hold for all numbers, not just the counting numbers, then we can prove some subtle observations about numbers.

For instance, beyond the counting numbers we like to believe that there is a number zero, denoted 0, which is an "additive identity." That is, it is a number with the property that $a+0=a$ for all numbers a. (And by the believed commutativity property of addition it follows that $0+a=a$ as well, for all numbers a.)

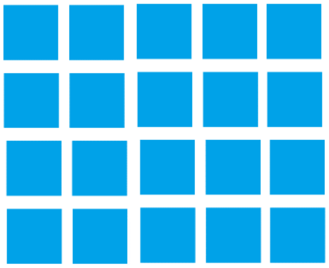
b) Prove that there are not two numbers that both deserve to be called zero. That is, if 0 is a number with $a+0=a$ for all numbers a, and $0'$ is another number with $a+0'=a$ for all numbers a, then it must be the case that $0=0'$. (That is, they are the same number.)
HINT: Compute the value of $0+0'$ two ways.

c) We like to believe that there is a number a such that $6+a=0$. (This number is denoted "-6.") Prove that there are not two such numbers. That is, prove that is $6+a=0$ and $6+b=0$, then it must be the case that $a=b$. Use only the commutativity and associativity rules of addition to establish this.

HINT: Compute $(a+6)+b$ two ways.

HOMEWORK SOLUTIONS:

Question 1: Here is a picture of 20 objects arranged in a four-by-five array.

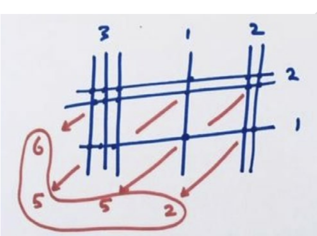


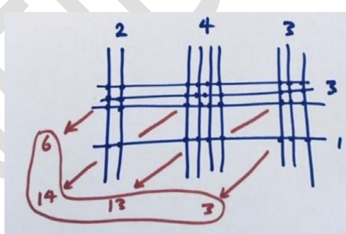
To distribute the 20 objects equally among five students, give each student a column of objects. As there are four rows in this picture, each column contains four objects.

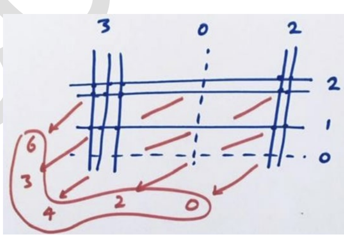
But also, each row is a group of five objects. As there are four rows in the picture, we can find four groups of five among these 20 objects.

In general, if a is a multiple of b, say, $a = kb$, draw a k-by-b array of objects. There are k rows. This is the number of groups of b one can find in the picture and also the number of objects each student receives if given a column of objects.

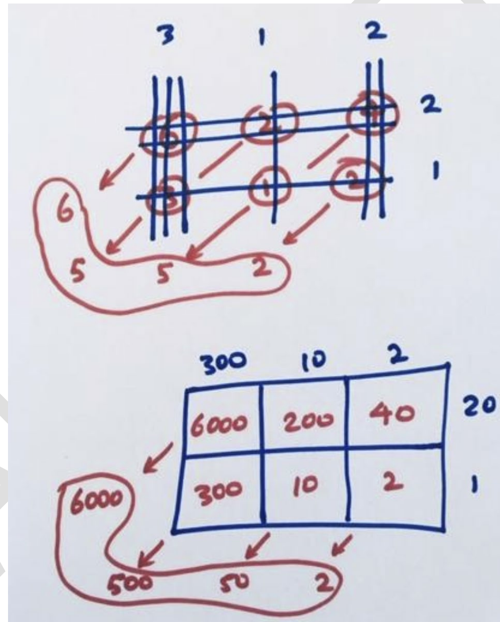
Question 2: a) One has to be careful with carries and zero digits. The answers are, respectively, 6552, 7533, and 63420.







b) Brief answer: Consider 312×21 again. We see that the "line method" of multiplication matches the area model, except it ignores the place value of each digit. The area model has matching units, tens, hundreds, etc, appearing on each diagonal. So adding along the diagonals in the line model is appropriate for the units, tens, hundreds, etc, place values too.



Question 3: If we go "-2" units to the left (that is two units to the right)and 3 units to the right, then the parabola method asks us to find the y-intercept of the line connecting two points on the same right side of the vertical axis, namely, the points $(2,4)$ and $(3,9)$. In general, the equation of the line that connects (a,a^2) and (b,b^2) is $y=(b+a)x-ab$ with y-intercept $-ab$. This suggests that $(-a)\times b=-ab$.

Similarly, this model gives $a\times(-b)=-ab$. (Here, we have the y-intercept of the line connecting two points to the left of the vertical axis: one a units to the left and the other $-b$ units to the right.)

And finally, $(-a)\times(-b)$ corresponds to y-intercept of the line connecting (b,b^2) and (a,a^2). One checks that this is ab.

Question 4:

a) For the right picture we see a diagram of two blocks next to three blocks all next to a diagram of four blocks. That is, we can see $(2+3)+4$.

By the same token, we can view the picture as a diagram of two blocks next to a diagram of three blocks next to four blocks. That is, we can see $2+(3+4)$.

It is the same diagram, so these two expressions must be equivalent.

b) By the commutativity property of addition we have that $0+a=a$ for all numbers a. Choose $a=0'$. Then this reads: $0+0'=0'$.
By the same token, $a+0'=a$ for all numbers a. Choose $a=0$ this time to see $0+0'=0$.

We thus have $0=0+0'=0'$.

c) We have

$$(a+6)+b=0+b=b.$$

We also have

$$(a+6)+b=a+(6+b)=a+0=a.$$

It follows that $a=b$.

Assessment Questions

Consider including the following two questions in a quiz or an exam.

1. a) Use the area model to compute $\frac{x^6 - 1}{x^2 - 1}$.

   b) Is $102^{60} - 1$ a prime number? How do you know?

2. We like to believe that all numbers satisfy three multiplicative properties: For all numbers a, b, and c we have

   i) $ab = ba$

   ii) $a \times 0 = 0$

   iii) $a(b + c) = ab + ac$

   We also like to believe that there is a special number, denoted 1, which serves as a "multiplicative identity," that is, has the property that $a \times 1 = a$ for all numbers a.

   Prove that that there cannot be two multiplicative identities. That is, if $1'$ is another number with the property that $a \times 1' = a$ for all numbers a, then we must have $1 = 1'$.

Projects

1. Over the centuries, scholars struggled to identify the simple key beliefs that serve to explain the underlying structure of the arithmetic of numbers.

In the 1800s scholars finally identified a set of basic beliefs that seem to capture the basic arithmetical properties of all real numbers (and complex numbers too!). They called any system of "numbers" that satisfy these basic properties a *field.*

a) Look on the internet for the list of "field axioms." Does the set of real numbers satisfy these axioms? Does the set of rationals? Does the set of integers?
b) The property $a \times 0 = 0$, which we identified as a fundamental belief in this lesson, is not listed as a basic belief in the list of field axioms! Can you prove that this property follows as a logical consequence of the listed axioms?

2. The existence of a number deserved to be called "zero" was debated by scholars over the millennia. Is zero a counting number? For example, if I note that there are zero blue giraffes in the room with me right now, is that because I counted zero blue giraffes or because I simply observed a lack of blue giraffes?

Research the story of mankind's debate and gradual acceptance of zero as a valid number.

CCSSM Upper-level Content Standards Addressed

CCSS.MATH.CONTENT.HSA.APR.A.1
Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.