**MATH CIRCLE SESSION**

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# FREAKY FIXED POINTS

**OVERVIEW:**

Imagine two pieces of paper stacked on top of one another. At present each point on the top piece of paper sits directly above its matching point on the bottom piece.

Now crumple the top piece of paper into a ball and throw it back on top of the bottom layer. Astoundingly, there is sure to still be at least one point in that top crumpled piece sitting directly above its matching point on the bottom!

Or take two maps of the U.S., one smaller than the other. Toss the small map somewhere onto the large map and you'll be sure to find on point in the U.S. in exactly the same spot on both maps.

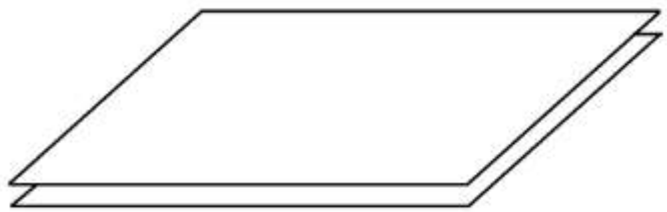
And watch out when you stir your morning drink – in an ideal mathematical world, at least one point of liquid is sure to return to exactly the same location it was at before stirring.

Let's have fun playing with clever "ABC triangles" to prove the existence of freaky fixed points.

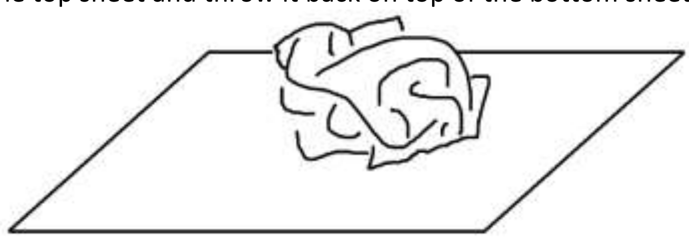
## THE ACTIVITY

## PART I: WHICH OF THE FOLLOWING MIGHT YOU BELIEVE?

1. Take two sheets of paper and place one directly on top of the other. At present each and every point on the top sheet sits directly above its matching point on the bottom sheet.



Now crumple the top sheet and throw it back on top of the bottom sheet.

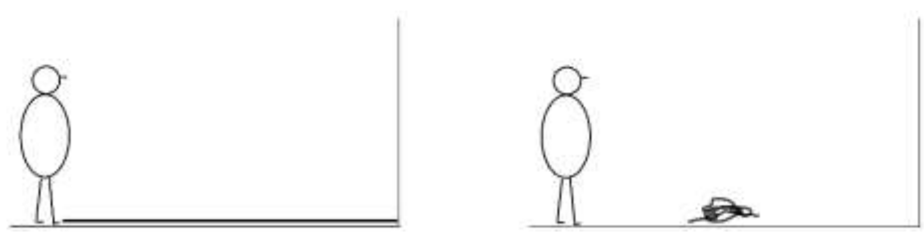


Do you think there could be a point in the crumpled top piece of paper still sitting directly above its matching point on the bottom piece of paper?

Do you think such a point must exist?

2. Let's consider a one-dimensional version of this scenario.

Lay out a rope on the floor directly in front of you stretching out to the wall. At present, each point on the rope is some distance from your feet.



Now pick up the rope, crumple it, and throw it back down in front of you.

Do you think there could be some point in the crumpled rope the same distance from your feet as it was before the rope was crumpled? Do you think there must be such a point?

Crumpling is too hard to think about.  How about  …?

**3.** Imagine I threw a small copy of a large map of the U.S. onto the large map. Do you think there is a location in the U.S. which sits at the same common point on both maps?


**4.** Let's take this down a dimension too.

Here's a picture of a ruler and a scaled-down picture of the same ruler. Is there vertical line on the page at which both rulers read the same number?

Must there always be such a location?

## Some Possible Steps towards Answers:

Question 4 might seem most believable.

> Let's compare the location of each number on the big ruler with its matching location on the small scaled copy of the ruler. We can think of each point in the big ruler as moving to a new location.
>
> The left end of the large ruler, the point $0$, "shifts" somewhere to the right in the small scaled picture and the point $12$ moves somewhere to the left.
>
> In fact, all the numbers near the left end of the large ruler shift rightwards and all the numbers on its right end shift leftwards. There must be some intermediate point that does not shift at all. This non-shifting point is a location with a common reading on both rulers. (It looks that that number is $7$ in the first picture and about $2.4$ in the second picture.)

This reasoning might suggest question 3 is believable: we have both left/right and up/down shifting going on, so maybe there is sure to be a point that shifts in neither direction and so stays fixed in location for both maps? Hmm.

Perhaps question 2 is believable too. Imagine a big ruler on the ground with positon $0$ at your feet and position $12$ at the wall. The left end of the rope, the one at your feet, starts at position $0$ and moves somewhere to the right in the crumpled rope. The right end of the rope, at position $12$, moves somewhere to the left. Maybe there must also be some intermediate position that moves neither left nor right, that is, is at the same distance from your feet in the crumple as it was before it was crumpled?

Question 1 still seems too hard to think about.

**Some jargon:**
A point that does not change location after some transformation or action is called a *fixed point* for the transformation. Each of the four puzzles above asks about the existence of a fixed point.

## PART 2: A COMPLETE CHANGE OF TOPIC

Here is a diagram of triangles. The outer vertices of the diagram are each labeled either A, B, or C. Your job is to label the inside vertices each, again, either A, B, or C, but there is one restriction: I don't want any fully labeled ABC triangles to appear. (That is, no triangle in the completed diagram is to have one of its vertices labeled A, another labeled B, and its third labeled C.)

Can you the label this diagram so that no ABC triangles appear?

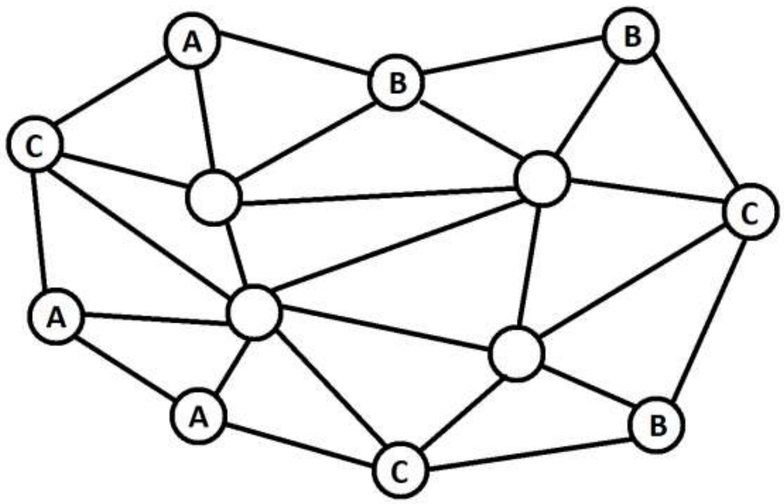



*Diagram 1*

Okay. How about this diagram?

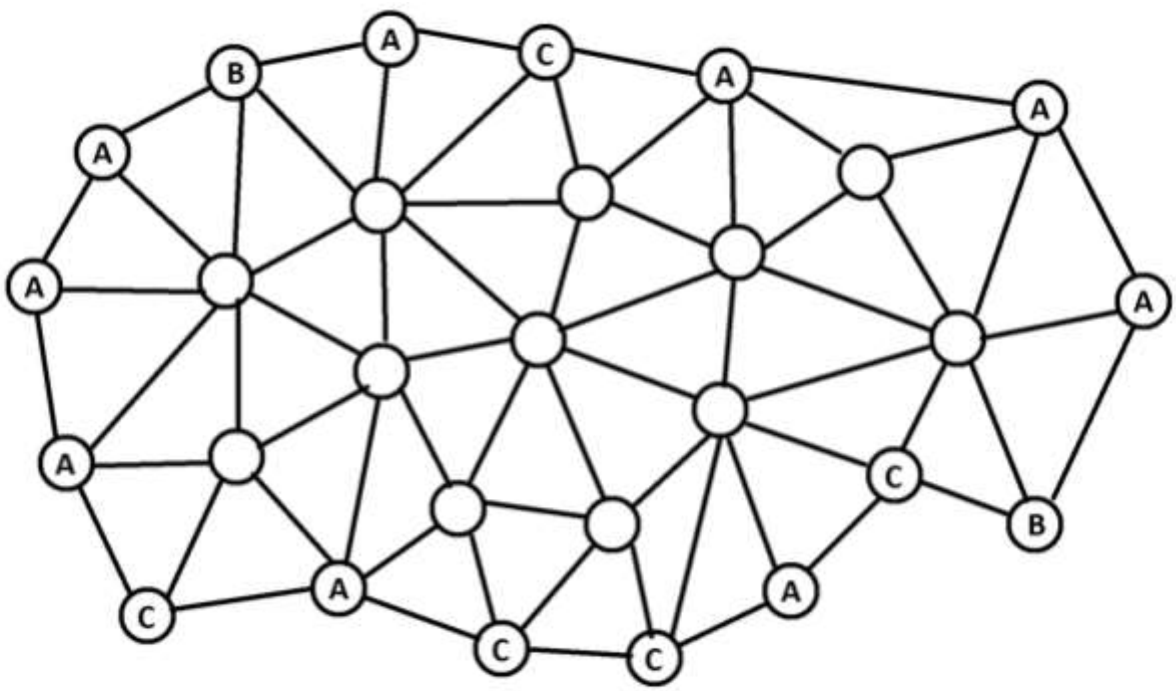



*Diagram 2*

If frustration is not too high, label this diagram too.

How do I know that you are forced to have at least seven fully-labeled ABC triangles no matter what you try to do? (Actually just quickly throw in some labels. Am I right? Do you have indeed have seven or more fully-labeled ABC triangles?)

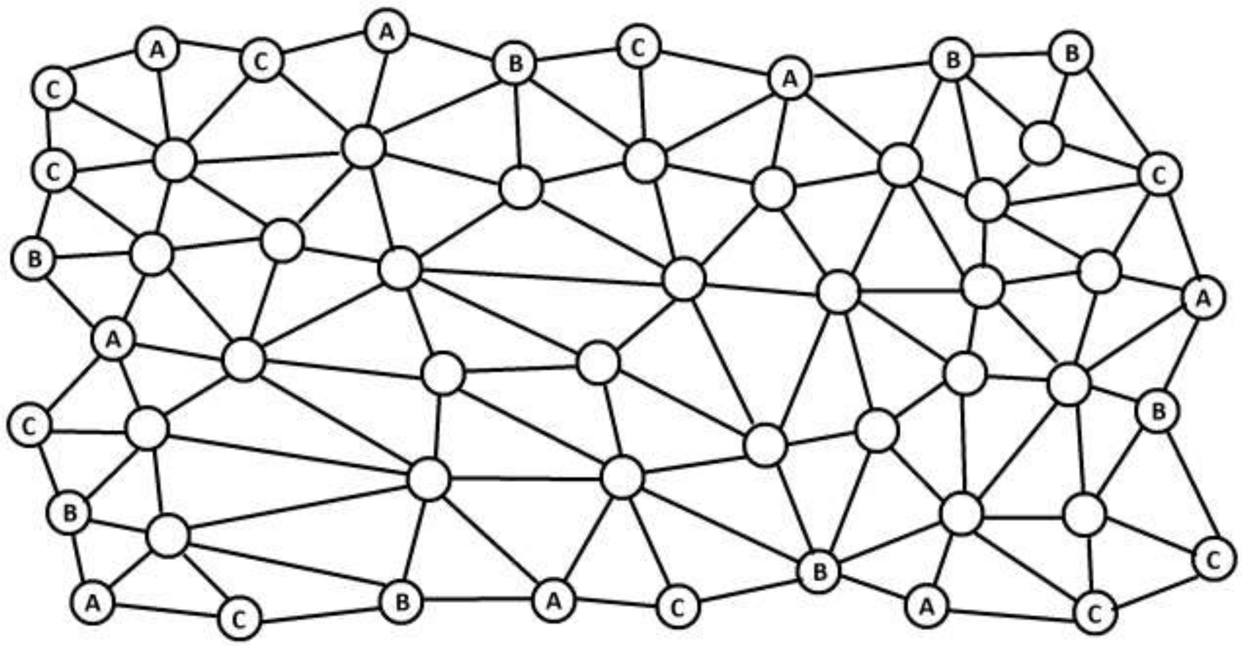
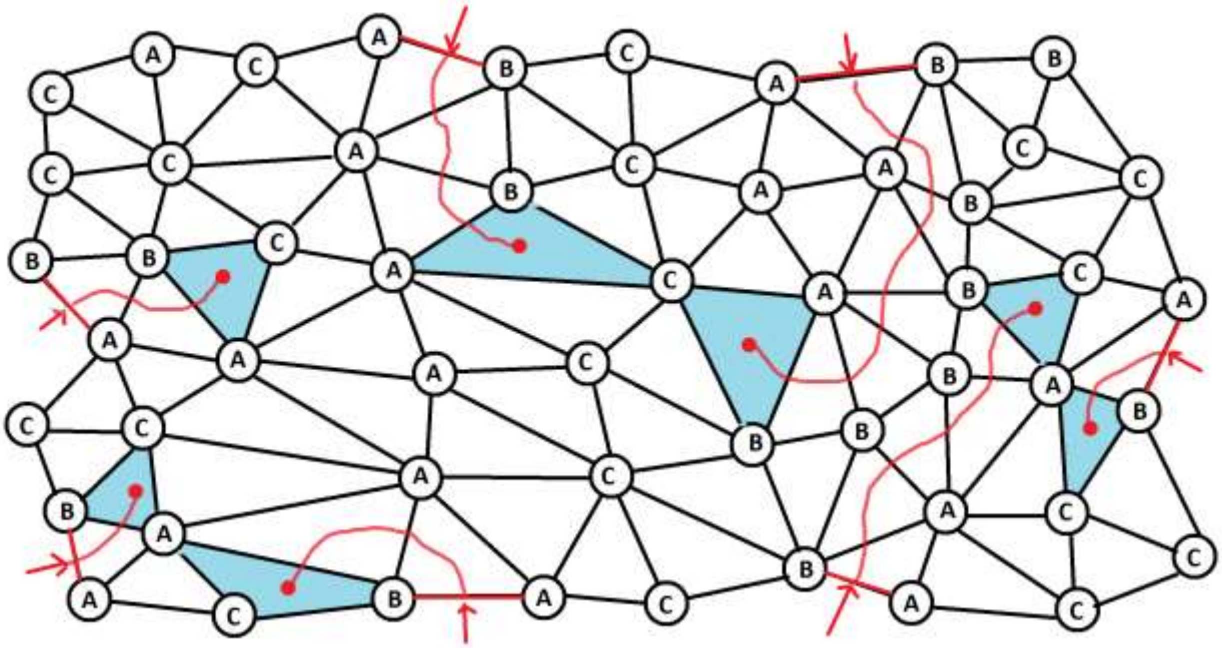



*Diagram 3*

## Explaining why ABC triangles are unavoidable:

Let's play with the third diagram. I've had a try at filling in the labels below.

Imagine that the diagram is a floor plan of a palace with nothing but triangular rooms. Every edge is a wall, except the edges that are labeled with an A and a B: all AB edges are doors to walk through. This palace has seven outside AB doors.



If we enter the palace though an outside AB door we'll be in a triangular room. Either it is an ABA room or an ABB room with a second door to walk through, or it is an ABC and we're stuck. If we have another door to walk through, let's walk through it and be in another ABA, ABB, or ABC room.

In this way each outside AB door leads us through a path of rooms. Eventually the path must stop and this happens only when we find ourselves in a room with no second AB door. That is, we'll be stuck in a fully labeled ABC room.

As there are seven outside AB doors, each must lead to a different ABC room. (Why can't two different paths lead to the same AB room?) Thus there must be at least seven fully labeled triangles in any labeling of Diagram 3.

## HANG ON! A Counter-Example!

This next diagram, Diagram 4, has four outside AB doors and so any labeling of the diagram should contain four fully-labeled ABC triangles according to the previous argument.

However … I bet you can find a labeling of the diagram that gives NO fully-labeled ABC triangles!

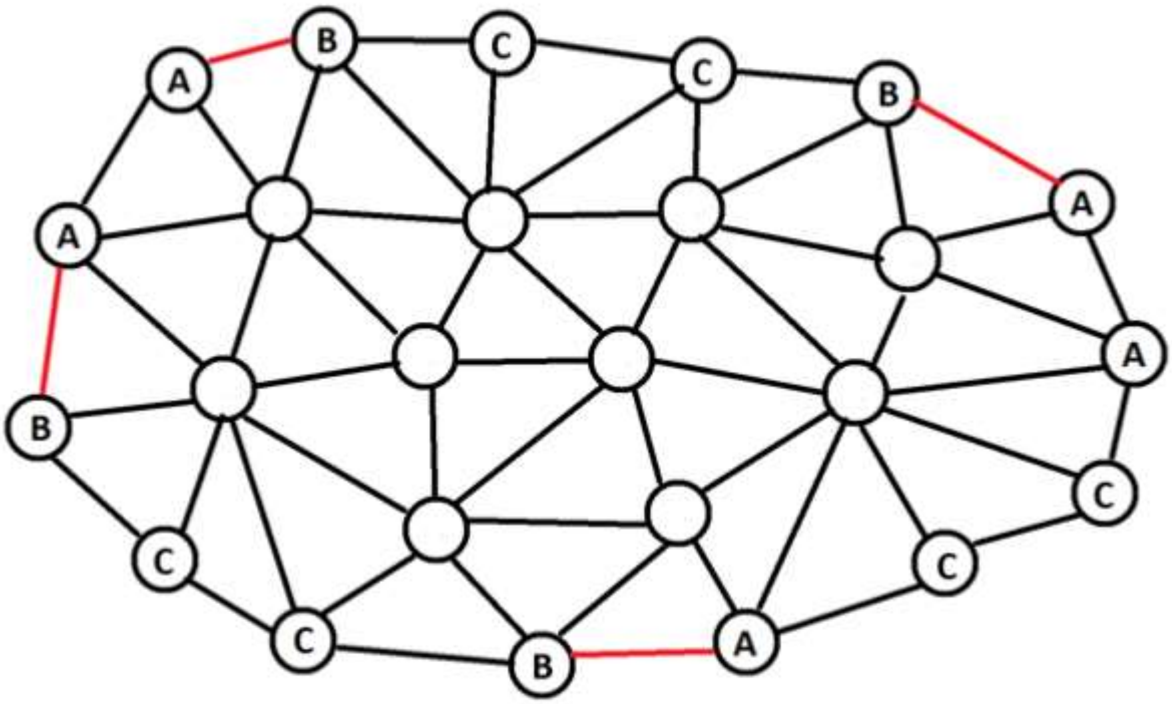



*Diagram 4*

What then was wrong with our previous argument?

**Answer:** Did you notice in your labeling of Diagram 4 that any path through one outside AB door leads you through the palace and back outside through another outside AB door?

There is nothing wrong with our previous argument other we need to attend to one more detail.

> Why must each path in Diagram 3 terminate inside the palace (and hence land in an ABC triangle) and not head outside through a second AB door?

If one enters the palace through an outside AB door with label A to one's right and label B to one's left, then every door one passes through thereafter has the same orientation. (Look at the paths we drew in a previous picture.) In Diagram 3, all the outside AB doors are oriented this same way. But approaching them from inside the palace, however, presents the wrong orientation for a path: from the inside perspective these doors have label A to the left and B to the right. It is thus impossible to exit the palace through any of these seven doors. It must be then that all seven AB doors for Diagram 3 give paths that terminate in the palace, that is, end in fully labeled ABC rooms.

**Some Comments:**

It is possible for a labelling of Diagram 3 to yield more than seven ABC triangles. Certainly each outside AB door leads to an ABC triangle, but there could be additional ABC triangles within the diagram that can't be reached from the outside. But note that, when standing in such an internal room, following a journey through its AB door takes you on a path that again terminates within the palace, that is, to another ABC room. This shows that all additional ABC triangles come in pairs.

For a general diagram …

> *If there are $n$ more outside AB doors of one orientation over another, then there are sure to be $n$ fully-labeled ABC triangles in any labeling of the diagram (and any triangles in addition to this come in pairs).*

(Each outside door with one orientation could be paired with an outside door of opposite orientation to create a path of rooms that enters the palace through one door and exits the palace through the other. This means that the remaining $n$ doors of the excess orientation must each give paths that terminate within the palace.)

There is nothing special about AB doors in this work. The same results hold for the counts of outside BC doors and for outside AC doors.

**EXERCISE:** Lulu drew some dots on the surface of a rubber ball and then connected pairs of dots with line segments to completely cover the surface with (spherical) triangles. She then randomly labeled each of the dots either A, B, or C. When done, she noticed that one of her triangles was fully-labelled ABC. Explain why, if she continues to look closely, she is sure to find a second fully-labeled ABC triangle on her sphere.

**EXERCISE:** Let $n_{ABC}$ be the number of ABC triangles in a labeled diagram (each with $1$ AB door), $n_{ABA}$ the number of ABA triangles (each with $2$ AB doors), $n_{ABB}$ the number of ABB triangles (each with $2$ AB doors), and $m$ the number of the remaining triangles (each with $0$ AB doors).

Let $e$ be the number of outside AB doors and $i$ the number of inside AB doors.

Explain why $n_{ABC} + 2n_{AAB} + 2n_{ABB} = 2i + e$.

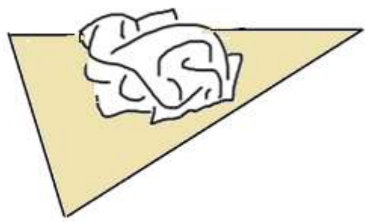
Use this to explain why if $e$ is odd (that is, there is an odd number of outside AB doors), then there must be at least one fully-labeled ABC triangle.

## PART 3: UNITING PARTS 1 AND 2

Let's now return to question 1 at the opening of this activity. But, for convenience, let's work with triangular, rather than rectangular, pieces of paper to start with. (We'll use the fact that triangles have three sides to develop a system for labeling with three symbols.)

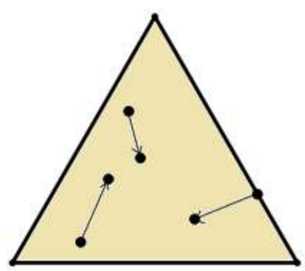
Here's our claim:

> *Take two congruent triangular pieces of paper with one initially lying on top of the other. Crumple the top piece of paper and place it back down on the bottom sheet of paper.*
>
> *As long as the crumpled paper lies directly on the bottom sheet (that is, no portion of the crumple extends beyond the boundary of the bottom sheet), then there is sure to be at least on point in the crumpled paper lying directly above it matching point on the bottom sheet.*



**Proving this to be so:**
Each point of the crumpled paper is likely to be sitting above a point different from its location before being crumpled.

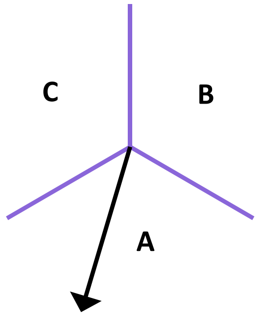
Actually, think of crumpling the top sheet of paper as a dynamic action that takes each point on the bottom sheet of paper and shifts it to a new location as given by the point it now sits above in the crumple.

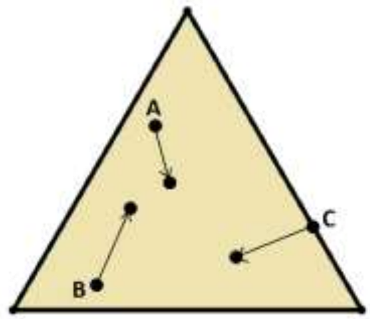


*The "motion" of points under crumpling*

Let's label each point on the bottom sheet either A, B, or C by the direction it shifts according to the key:

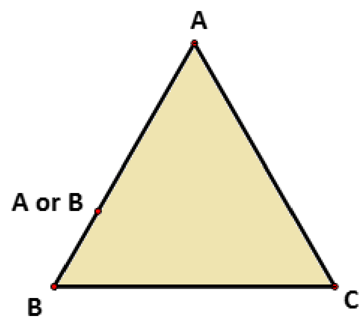
> B: all points that shift up to and including $120^\circ$ to the right of the vertical.
> C: all points that shift up to and including $120^\circ$ to the left of the vertical, as well as the points that shift vertically.
> A: all points that shift in any of the remaining directions (as per the black arrow in the figure, for example).



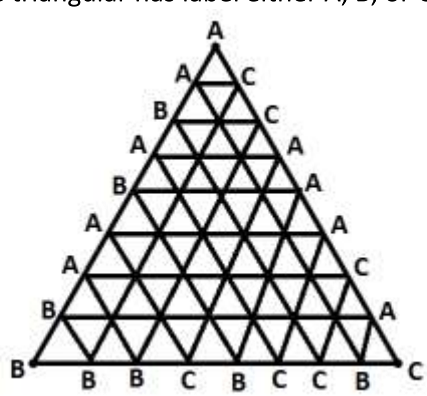

The only points we can't label are the ones that don't move. But if we find such a point, then we have what the claim is asking for – a point on the crumple sitting directly above its matching point on the bottom sheet of paper! In this case, if we find such a point, we need not do any more work nor follow the rest of the proof. (But we'll carry on writing the proof, however, for as long as possible assuming we don't ever stumble upon a fixed point.)

As each vertex of the triangle can only move inwards, they must receive labels A, B, and C as shown.




Any point on the edge with vertices A and B can only move in the direction of A or B and as will be so labeled. Similarly, any point on the edge with vertices B and C will be labeled B or C, and every point on the edge with vertices A and C will be labeled A or C. An interior point of the triangle can be labeled either A, B, or C.

Subdivide the triangle into a large number of smaller triangles as shown, each of area less than $0.1$ square units say. Each point in the triangular has label either A, B, or C.



We observe:

*The number of outside AB edges is odd.*

**Reason:** Outside AB "doors" can only appear on the left edge of the big triangle: the side with one vertex labeled A and one vertex labeled B.

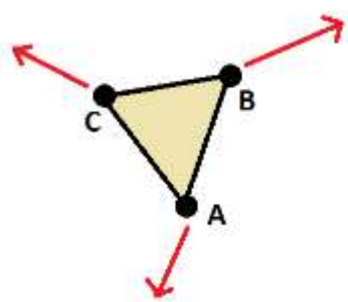
Slide your finger down the left edge of the triangle, starting at the top vertex A and ending at the left vertex B, and say the word "switcheroo" each time the string of labels you encounter change from being As to Bs, or change from Bs to As. As we start with the label A and end with the label B, you must say switcheroo an odd number of times. Also, you say the work switcheroo each time you finish traversing an AB edge. Thus there are an odd number of AB edges in all.

So with an excess of AB edges of one orientation over another, the labeling of the entire triangulation must contain at least one fully labeled ABC triangle of area no more than $0.1$ square units of area.

In the same way, by using a finer triangulation we can find an ABC triangle of area no larger that $0.01$ square units, and another of area no larger than $0.001$ square units, and so on.

We can find an infinite sequence of fully labelled ABC triangles of areas decreasing by factors of at least ten.

**An intuitive finish to the proof:**
Pick one of these fully labeled ABC triangles, one smaller than the size of an atom. Think about what is happening to its three corners: they are moving in three different directions.




This surely can't be the case for smaller and smaller triangles. There has to be some point that is not moving at all.

**A very technical scary finish to the proof for the purists:**
We have just found an infinite sequence of fully labeled triangles of areas decreasing by factors of at least ten. Denote their vertices as follows (matching the labels of those vertices):

$$A_1B_1C_1, A_2B_2C_2, A_3B_3C_3, \ldots .$$

Divide the large triangle into four quarters. One of those quarters contains infinitely many of the points $A_1$, $A_2$, $A_3$, ….

Divide that quarter triangle into quarters. One of its quarters contains infinitely many of the points $A_1$, $A_2$, $A_3$, .....

Divide that quarter triangle into quarters. One of its quarters contains infinitely many of the points $A_1$, $A_2$, $A_3$, .....

And so on.

This means that we can find a subsequence of the points $A_1$, $A_2$, $A_3$, …. with one point in a particular quarter of the triangle, the next point in a particular quarter of that quarter of the triangle, the third point in a particular quarter of the quarter of the quarter of the triangle, and so on. Since we can looking at finer and finer quarters of quarters of the triangle, we can see that this subsequence of points converges to a particular point $P$ in the triangle.

Since each point $A_i$ comes with its friends $B_i$ and $C_i$, each getting closer and closer to $A_i$ as $i$ increases, it follows that the matching subsequences of $B_1$, $B_2$, $B_3$, … and $C_1$, $C_2$, $C_3$, … converge to $P$ as well.

Now ask: In which direction does $P$ move?

Since each point $A_i$ moves downwards and $P$ is the limit of a sequence of points labeled A, it must be that $P$ s motion is a limit of downward motions. (More precisely, let $\theta_i$ be the angle from the horizontal the point $A_i$ moves. We have that $-150^\circ < \theta_i < -30^\circ$. Let $\theta$ be the limit of the $\theta_i$ s that match the points $A_i$ in our subsequence. Then $\theta$ is the direction $P$ moves and $-30^\circ \le \theta \le -150^\circ$.)

Since each point $B_i$ moves up to the right and $P$ is the limit of a sequence of points labeled B, it must be that $P$ s motion is a limit of upward/rightward motions.

Since each point $C_i$ moves up to the left and $P$ is the limit of a sequence of points labeled C, it must be that $P$ s motion is a limit of upward/leftward motions.

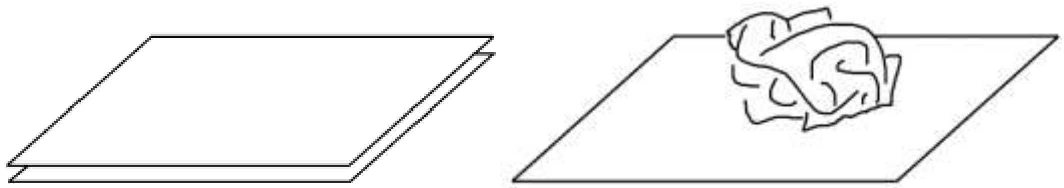
The only direction of motion that can be the limit motion of these three directions is zero motion. If we haven't found a fixed point anywhere in this proof so far, then $P$ is one for sure!

We have now proven that in any crumpled triangular paper, there is, for sure, at least one point in the crumple sitting directly above its matching point on the bottom sheet.

**Comment:** This proof relies on "continuity" – that points that start close to each remain close to each other. This means that we are assuming that the paper is never ripped in the crumpling process. (A tear separates close points.) If we allow for tearing, then there is no guarantee that fixed points will exist.

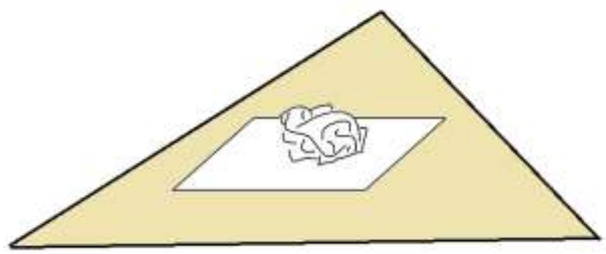
## PART 4: FINISHING DETAILS

**Rectangular paper?**
Our first question originally worked with rectangular paper.



We proved that crumpled triangular paper has fixed points. Must crumpled rectangular paper do so too?

Imagine the rectangular sheets of paper glued inside a large triangular sheets of paper. Crumple the top large triangle, and place the crumpled paper somewhere on the rectangle within the bottom triangle.



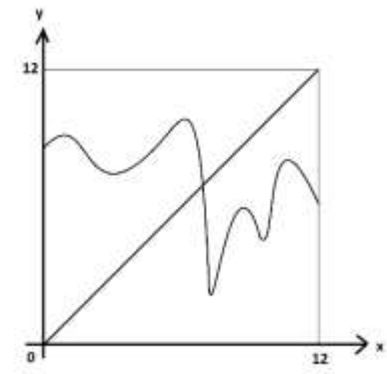
We know that there is a point somewhere in the crumple directly above the its matching point in the triangle. But this fixed point is actually within the rectangle. So ignoring the extra flaps of paper making the full triangle, there is a point in the crumpled rectangle of paper sitting directly above it matching point in the bottom rectangle.

**Shrunken Maps?**
Proving that fixed points exist for small copies of maps placed on large ones follows exactly the same line of reasoning as presented here for crumpling.

(Actually, it is easier to visualize how the proof works with this type of example. Perhaps work through our reasoning again but imagining a small triangular map placed somewhere on a large one?)

**Crumpled Rope?**
Each value of $x$ between $0$ and $12$ corresponds to a point on the outstretched rope a distance $x$ units from your feet. Let $f(x)$ be the distance from your feet that point now is in the crumpled rope.

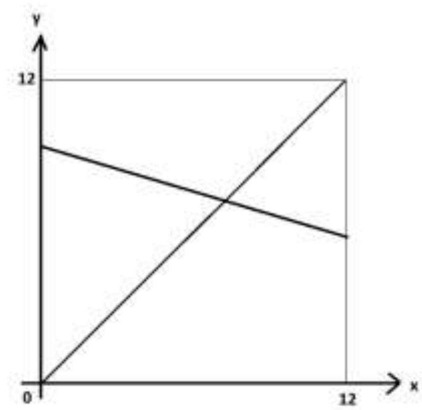
We have $f(0) > 0$ (the left end of the rope moved further away from your feet) and $f(12) < 12$ (the right end of the rope mover closer to your feet). If we plot a graph of $y = f(x)$, we get a continuous plot (the rope is never broken, we assume) that looks something like this:




If we trace the plot from left to right, the curve starts above the diagonal line $y = x$ (because $f(0) > 0$) and ends below the diagonal line (because $f(12) < 12$). There must be some location where the two curves $y = f(x)$ and $y = x$ intersect. This point $x$ is a location such that $f(x) = x$. That is, it represents a point on whose distance from your feet after crumpling is the same as its distance before crumpling. We have a fixed point.

**Shrunken Rulers?**

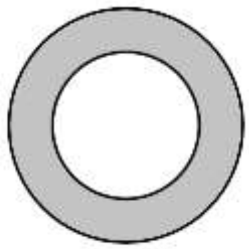
Follow the same idea as the crumpled rope. This time, since our "crumpling" of the ruler is now just linear scaling, the plot of $y = f(x)$ is just a straight line graph.



**Background:**
We have established what mathematician's call the Brouwer Fixed Point Theorem, at least, in dimensions one and two. (Luitzen Brouwer was a Dutch mathematician, 1881-1966.) The theorem says that any continuous map from a line segment to itself and any continuous map from a connected region in the plane (possessing no holes) to itself must each possess at least one fixed point: that is, there is sure to be at least one point that does not move under the mapping.

The theorem actually works in all dimensions. Some people like to interpret the three-dimensional version of the theorem as follows: *After stirring a cup of coffee, at least one point of liquid is sure to return to its original location.* (But there are problems with this interpretation. For example, liquid coffee is composed of discrete molecules and the theorem applies to a continuous range of points in space. Also, the spoon for stirring may separate points, causing the mixing not to be continuous.)

**Exercise:** Why does Brouwer's Fixed Point Theorem not apply to the this shape?



The proof of the two-dimensional Brouwer Fixed Point Theorem we presented here is due to German mathematician Emanuel Sperner (1905-1980). The lovely result about labeled triangulations and the existence of fully-labeled ABC triangles is today known as Sperners' Lemma.

The internet, of course, offers all one could possible want to learn – and more – about these results and these gentlemen.

## COMMON CORE CONNECTIONS:

**Mathematical Practice Standards**
Every math circle activity, by very definition, engages participants in the first, and most important, Mathematical Practice Standard:

> **MP1:** *Make sense of problems and persevere in solving them.*

And, because of the collaborative nature of math circles, the third practice standard is in the fore as well:

> **MP3:** *Construct viable arguments and critique the reasoning of others.*

The interplay between labeling and the geometry of fixed points certainly attends to:

> **MP2:** *Reason Abstractly and Quantitatively.*

One can even argue, especially with the open problems in part 2 of this activity, we must also attend to:

> **MP5:** *Use appropriate tools strategically.*

**Content Standards**
One can (and I personally have) used Brouwer's Fixed Point theorem to motivate the Common Core State Standards on transformational geometry (which is the basis for developing geometric thinking in these standards). For example, placing small maps on large ones and looking at scaled copies of rulers placed next to larger ones provides an intriguing problem motivating the desire to study dilations (and rotations and translations).

Instead of crumpling a piece of paper, where are the fixed points under a simple rotation? A reflection? (Do translations have fixed points?)

The fact that we talk about "where points move to" develops the necessary language and thinking to describe geometric mappings and translations.

In short, discussing the Brouwer Fixed Point Theorem sets the stage and a natural context for the content standards in the geometry cluster:

> **8.G.A** *Understand congruence and similarity using physical models, transparencies, or geometry software.*

Sperner's Lemma has no connection to the content standards (but it is a whopper of an idea for MP1!).

## HANDOUTS

Each point is to be labeled with A, B, or C.
Can you the label this diagram so that no ABC triangles appear?

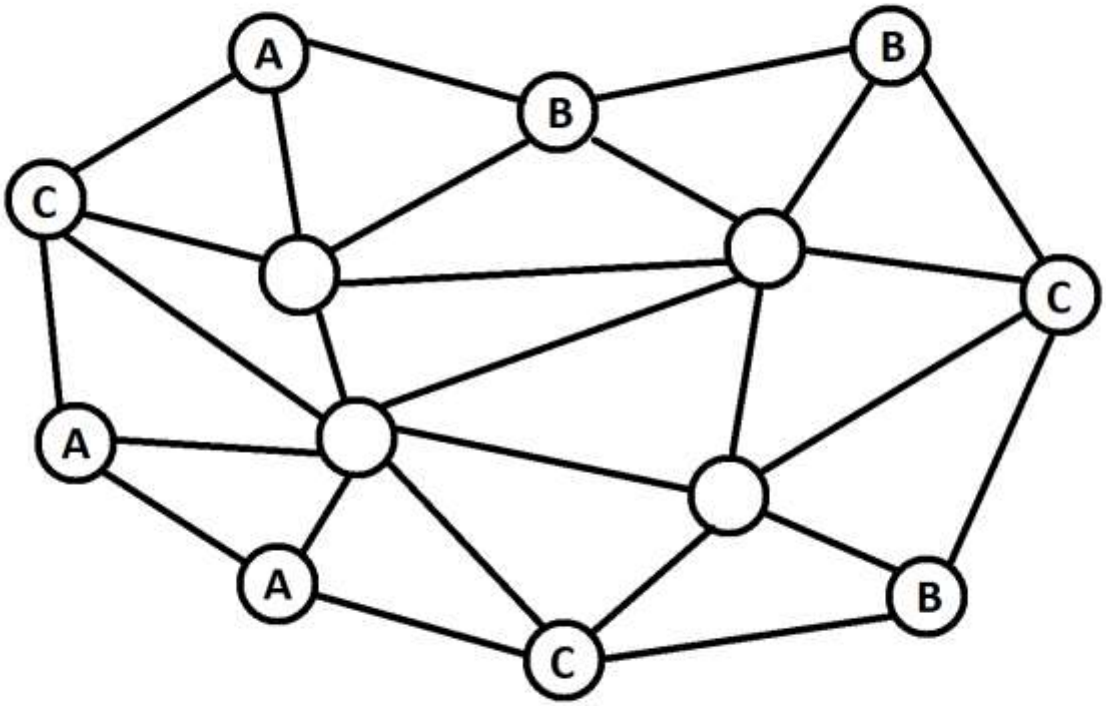



*Diagram 1*

Okay. How about this diagram?

Again each point is to be labelled A, B, or C and we don't want any fully-labeled ABC triangles to appear.

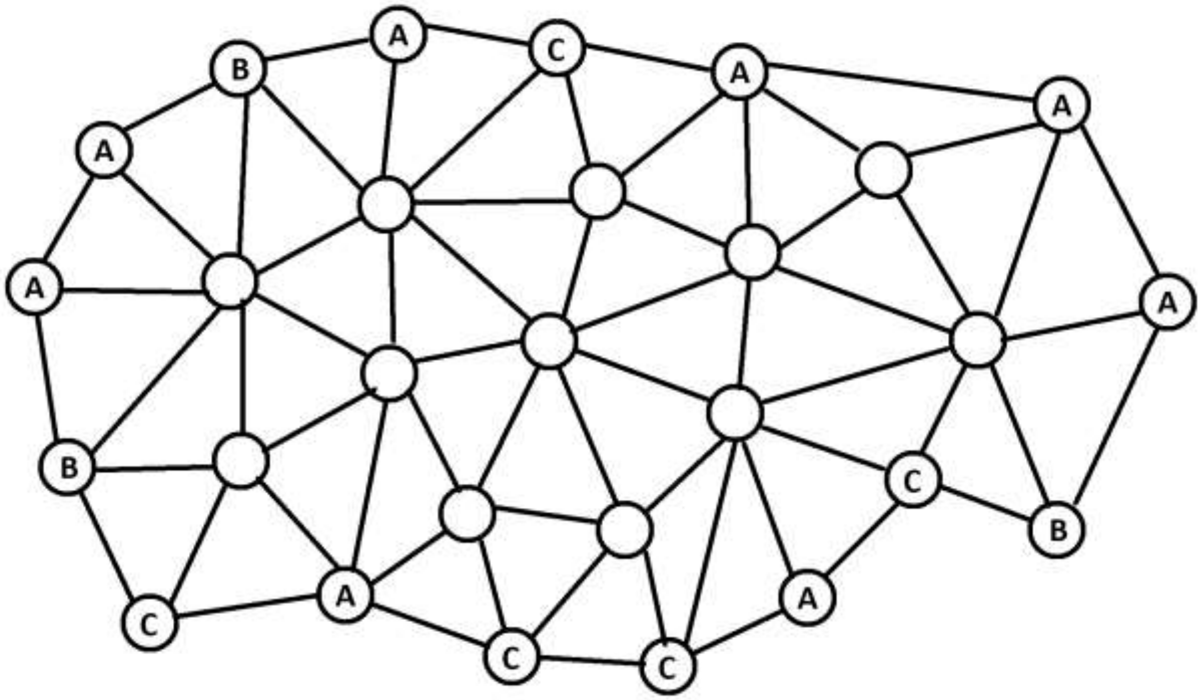



*Diagram 2*

Okay, I am feeling generous. I will allow for up to SIX fully-labeled ABC triangles in this next diagram, but no more. Can you label each vertex either A, B, or C so that no more than six fully-labeled ABC triangles appear?

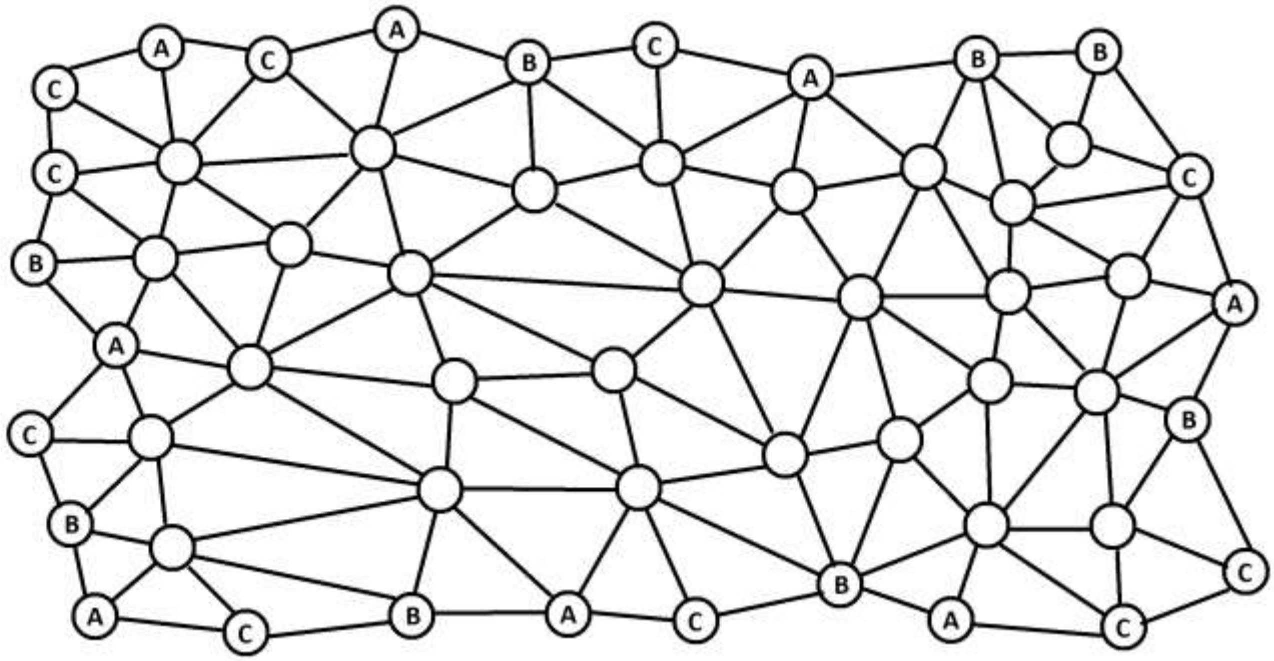


*Diagram 3*

No fully-labeled ABC triangles?

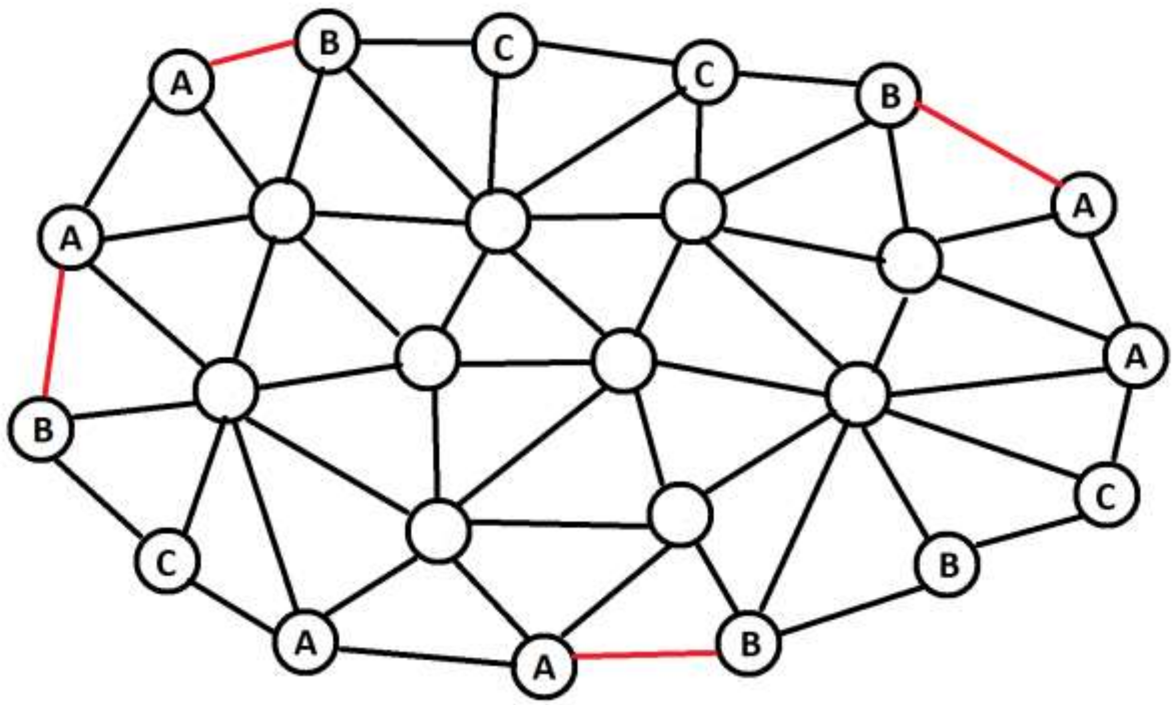



*Diagram 4*