MATH CIRCLE SESSION

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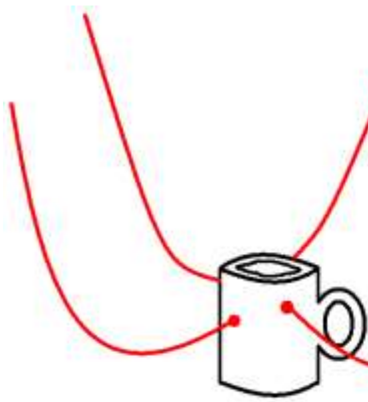
TEACUP TWISTS

and impossible braids

THE TEACUP PUZZLE:

Hold a teacup up in the air in the center of a room and have friends tape four or five strings from the cup to various points about the room. (Just two or three strings make too easy a puzzle.) Be sure to leave plenty of slack in the strings.



Rotate the teacup one full turn, 360°, tangling the strings in the process.

From this point on, the cup is to be held fixed in space, never to move again!

You and your friends' job is to now maneuver the strings around the cup and untangle them. (The person holding the cup - you - will have to move your hands out of the way as folk bring strings up and over or down and around the cup. But the cup itself is not to move from its position in space, nor turn or tilt in any way.) Can you and the team untangle the strings?

TOWARDS SOLUTIONS:

You and the team might struggle with this task. Give it a good try, but if frustration levels become high try the following:

Give the cup another full turn IN THE SAME DIRECTION (720° *), tangling the strings even further!*

Again holding the cup fixed in space, maneuver the strings around the cup and untangle them. **This second task can definitely be done!** Absolutely do try it!

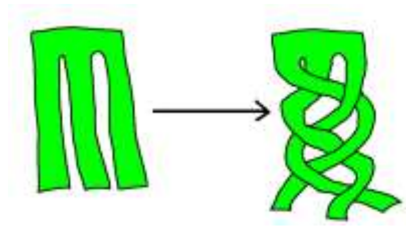
When ready, return to the one-full-turn puzzle and try it again.

If you seriously can't untangle the strings with just one full turn could it be that it is a mathematical impossibility? Can you prove it can't be done? (It would be mighty curious though if it is impossible to untangle a tangle from one full rotation but always possible to untangle a doubly-worse tangle from two full rotations!)

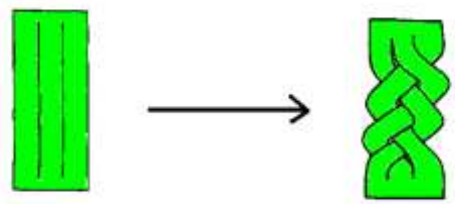
When ready for a break from all this, just let it be and move on to the next puzzle on the next page. We can come back to teacups later.

PLAYING WITH IMPOSSIBLE BRAIDS

To make a braid one usually starts with three parallel strands joined together at one end but kept loose at the other.



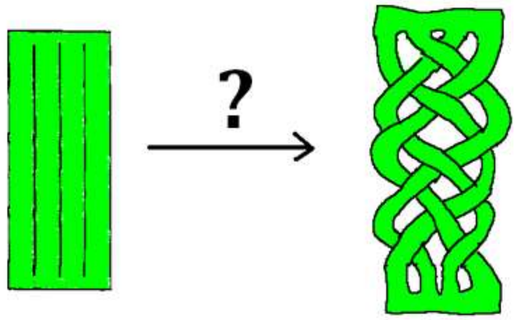
But why bother with the loose ends? Go ahead and make a braid with no free ends! It can be done!



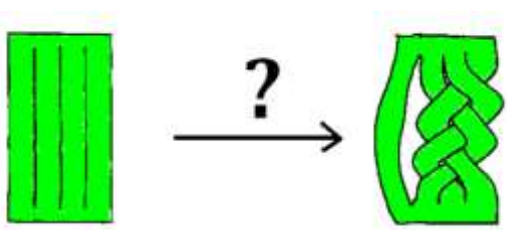
Here's me holding a no-free-end braid made from paper. Notice that the individual strands are relatively flat: there are no twists and the same one side of the paper faces outwards at all times within each strand.

Comment: Paper is very hard to work with – it is annoyingly inflexible! I recommend cutting two slits in a rectangle of felt and playing with that instead.

CHALLENGE:
Why stop at three strands? Can you make this no-free-end version of a four-strand braid? Try it!

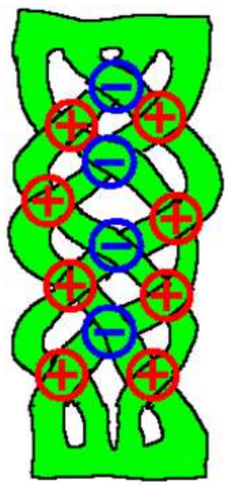


How about a three-braid with an extra strand along for the ride?



GIVING SOME THINGS AWAY:

Let's look at the four-strand braid. Notice that strands cross a total of 12 times, sometimes with the right strand crossing over the left (let's call these positive crossings) and sometimes with the left strand over the right (negative crossings).

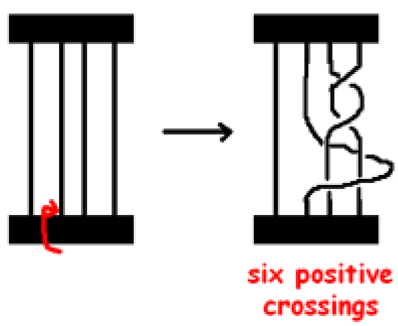


Index = +4

The diagram we hope to construct has four more positive crossings than negative crossings. Let's say it has index $+4$.

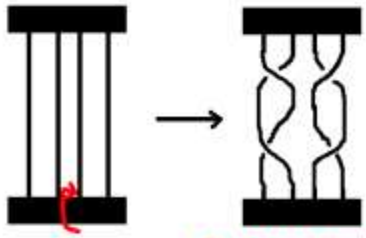
But look at the maneuvers we can perform on this system. There are essentially only four types of moves.

MOVE 1: *Push the bottom end of the felt through a slit to the left or to the right.*



We see that this move introduces six new crossings, all of the same sign. This means the index changes either by $+6$ or by -6.

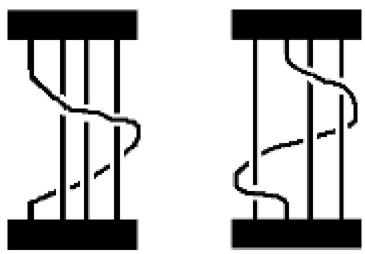
MOVE 2: *Push the bottom end of the felt through the middle slit.*



This move introduces four new crossings, but the total index does not change.

MOVE 3: *Pick up and move one strand around the base of the object.*

Here are two typical examples of this move.



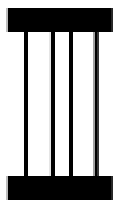
We see that the total index again changes by ± 6.

MOVE 4: *Rotate the bottom half of the felt half a turn.*

Such a move introduces six crossings all of the same sign and changes the index by $+6$ or by -6.

SO .. Is it possible to create the four-strand of index 4 ?

NO! We start with the figure of index zero:



and every move we perform on this object either keeps the index the same or changes it by six. Whatever we create from this starting position must be a figure with index a multiple of six. An object with an index of four is thus unattainable.

Comment: My photograph of a four-strand braid with no free ends is a cheat! I had to cut one of the strands to make it and hid the cut behind another strand.

CHALLENGE: This braid has index -6. Does this mean that the following construct is attainable?

If so, attain it. If not, what this time is the mathematical obstruction?

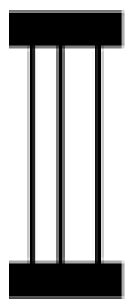
RETURNING TO TEACUP TWISTS

You are now in position to prove that it is impossible to untangle strings tangled by rotating a cup just one full turn (360°).

Let's focus on a cup with just three strings.

1. Explain why if it is impossible to untangle just three strings taped to a teacup under one full turn, it is thus impossible to tangle more than three strings taped to a teacup under one full turn.

Here is a schematic of three untangled strings tied to a teacup.



2. Following the work of the previous section, what is the index of tangle produced by one full turn?

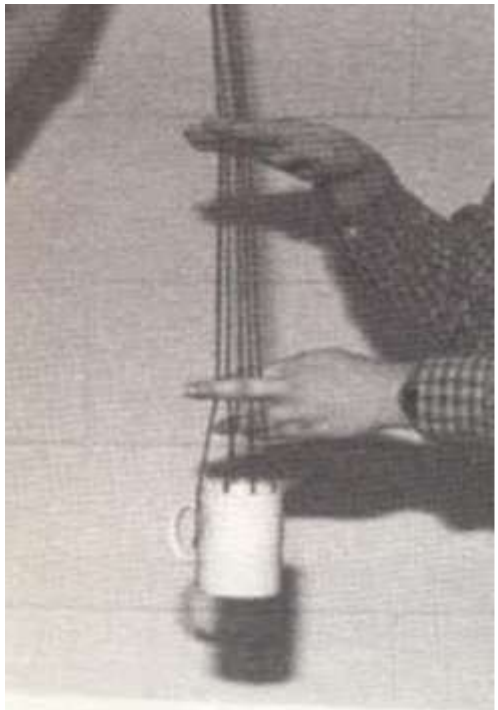
3. Following the previous section … Only moves of type 3 are permitted in trying to untangle the strings of a teacup. What is the index of any such move?

4. Explain why it is impossible to untangle three strings tied to a teacup that have undergone one full turn.

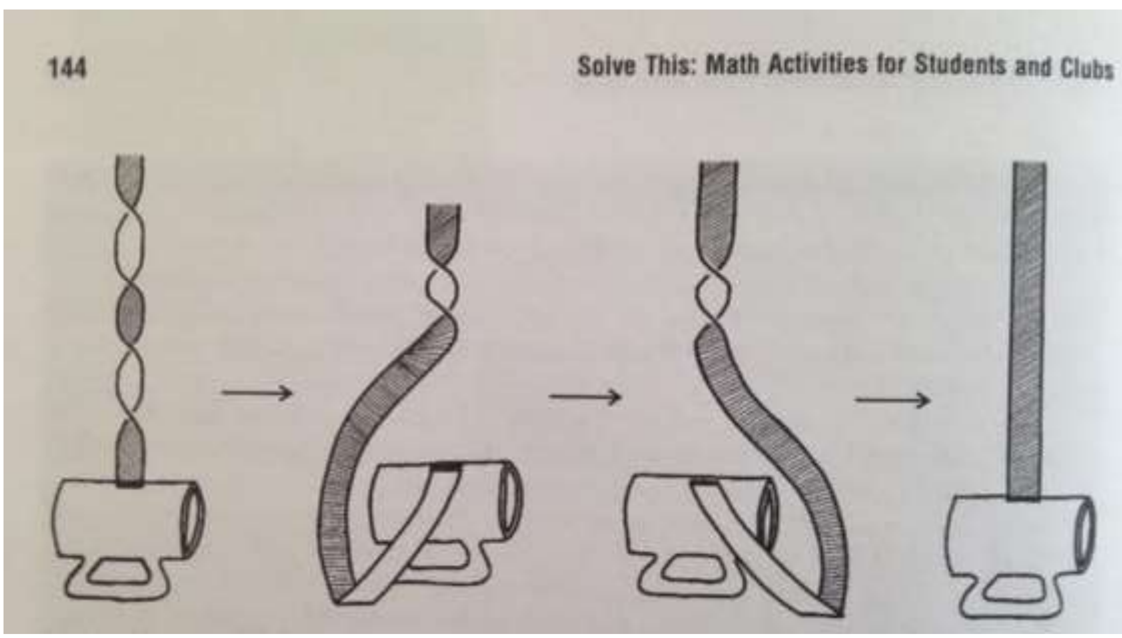
TWO FULL TURNS:

The way to prove that tangles from two full turns can always be untangled is to demonstrate it!

A good way to do this is to imagine all the strings lined up in a band, as shown in this next poor-quality photograph:



So we can think of the tangle from two full turns as the twist in a band from two full turns. And we can see this can be untangled by maneuvering it around the cup.



Try this with your belt! Two full twists in a belt can be undone by moving the belt around the buckle.

Comment: It is possible to untangle the strings of two-turn teacup by moving all the strings at once in this fashion. Or one can maneuver each string one at a time in this fashion.

A classic version of this phenomenon is called the WAITER'S TRICK. Here student Alex Alapatt shows that you can give a tray of drinks (or a box in this case) one full turn while standing still in place.

It is surprising that you can give the tray A SECOND FULL TURN IN THE SAME DIRECTION! One does this by moving the box over your head.

Notice how Alex returns back to his initial state after two full turns.

(Do try the waiter's trick. It is fun!)

Comment: Physicist Paul Dirac used this Waiter's Trick to demonstrate why, in quantum mechanics, angular momentum should come in half packets: two full rotations of a body about an axis return you to the initial state, not just one.

(So watch out when you next go out dancing. Be sure to twirl and even number of times lest you end up in a different quantum state!)

CHEAP FINAL PUZZLE: PUZZLER: Pick up a piece of string from a table top, one end of the string with your left hand and the other end with your right hand.

Now, without ever letting go of either end of the string, maneuver your arms and your body so as to eventually tie a knot in the string. It can be done!