THE MATHEMATICS OF SYMMETRY

Smart Phones, Frieze Patterns, Fractals, and More!

JAMES TANTON

www.jamestanton.com

www.gdaymath.com

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ON DEFINING SYMMETRY

The idea of symmetry seems to play a significant role in our human experience. Whatever symmetry is, our human sense of aesthetics seems to be drawn to it. We like regularity, pattern, predictability, reproducibility.

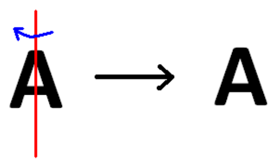
And we humans seem to be able to recognize symmetry when we see it, even if we haven't ever defined what symmetry is.

Consider, for example, the (capital) letters of the alphabet:

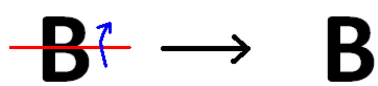
ABCDEFGHIJKLMNOPQRSTUVWXYZ

We would all agree, looking at this list, that some letters are more symmetrical than others.

For example, the letter A seems to have "bilateral symmetry" about a vertical line: Flip the letter A vertically and it still looks like the letter A.

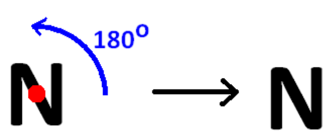


The letter B has bilateral symmetry about a horizontal line:



The same is true for the letters C, D, and E, and others.

Letters such as N, S, and Z possess a different type of symmetry: "180° rotational symmetry."



Letters such as O and X possess all these symmetries. Letters such as F, J, P, R none.

O X F J P R

It seems we can state when objects do and don't possess "symmetry." Can we give a definition as to what symmetry actually is?

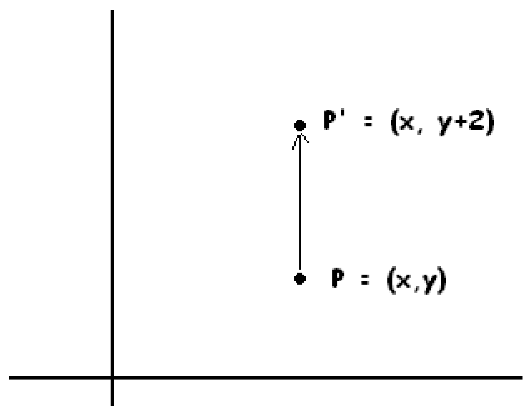
The thing to note is that we seem to be talking about motions in the plane that map points of a figure back onto the figure. So to capture the meaning of "symmetry" we need to first understand mappings in the plane.

MAPPINGS: SOME STANDARD EXAMPLES

Loosely speaking, a mapping, is a rule that shifts some/all points of the plane to possibly new locations in the plane.

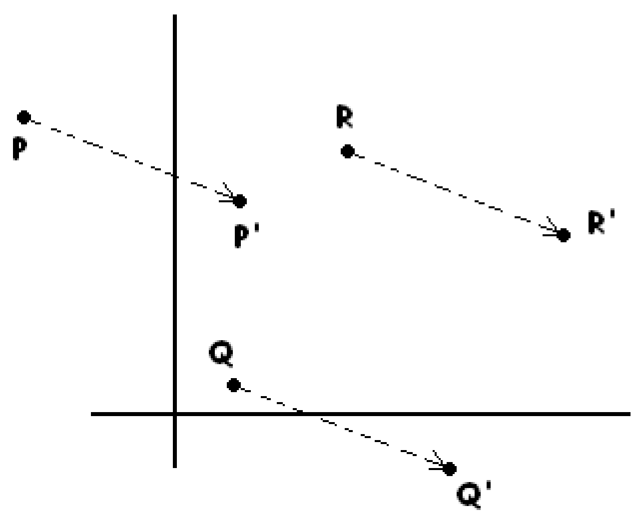
Some examples make this loose definition clearer.

EXAMPLE 1: The rule that shifts each point of the plane two units in the positive y -direction is a mapping.



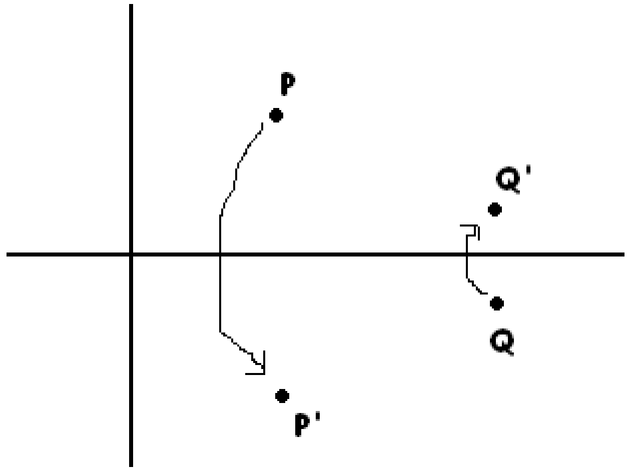
This is an example of a mapping called a **translation**.

In general … a translation is a mapping that shifts each point in the plane a fixed distance in a fixed direction.



Note that every point moves to a new location in this mapping.

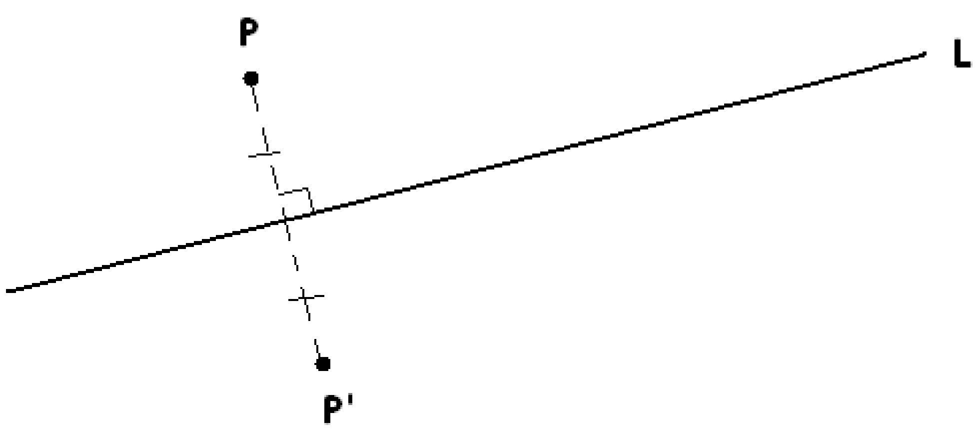
EXAMPLE 2: The rule that reflects each point about the *x*-axis is a mapping.



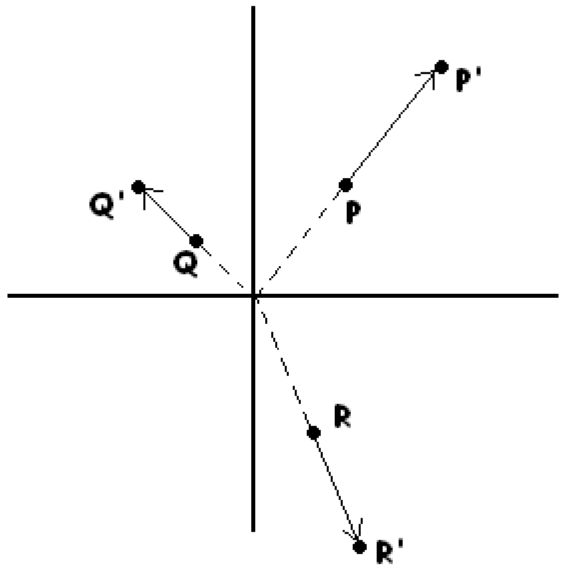
This is an example of what is called, surprise surprise, a **reflection**.

Note that points in the x-axis itself do not move under this mapping.

Writing down a precise definition of reflection takes some doing. In general, a reflection about a line L is the mapping that takes each point P not on the line and maps it to a point P' so that the line L is the perpendicular bisector of $\overline{PP'}$, and takes each point Q on the line L to itself.



EXAMPLE 3: The rule that "pushes" each point away from the origin to double its distance from the origin is a mapping.



This is an example of what is called a **dilation**.

Note that the origin itself does not shift in this mapping.

In general, a dilation about a point O in the plane with scale factor k (a positive real number) is the mapping that takes a point P, different from O, to the point P', so that O, P, and P' lie on the same ray with endpoint O and so that $OP' = kOP$. The dilation keeps the point O itself fixed in place.

Exercise 1: What does a dilation with scale factor $k = \frac{1}{2}$ do? Is the idea of "pushing away" from the point O still an appropriate description?

Exercise 2: Might it be meaningful to discuss a dilation with $k = 0$? Or one with a negative scale factor?

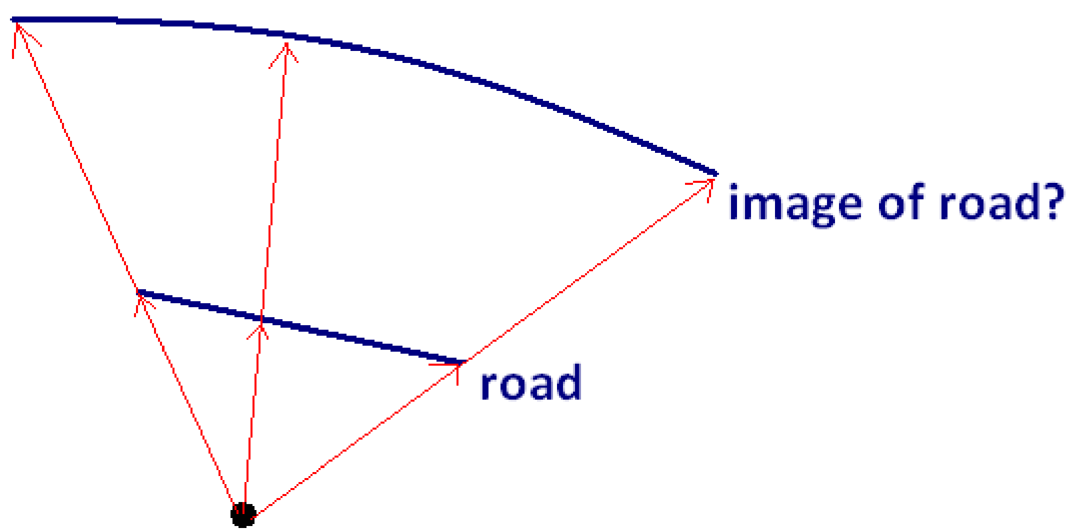
MYSTERY: MAPS AND SMART PHONES

When one opens the map feature on a smart phone one uses a "two finger swipe" to zoom in on the map:

Place two fingers on the screen and slide them apart.

The result is actually a dilation on the screen: Points are being pushed away from the midpoint of your two fingers, radially outwards! All the points to the left and right of the motion of your two fingers are being pushed left and right, all the points diagonal to the motion of your two fingers are being pushed diagonally outwards, and so on. Look closely!

MYSTERY: A dilation "pushes" all points radially outwards from a given point. And radial motion, intuitively, to me at least, should curve straight lines.



But when one zooms in on a smart phone map, all the straight roads in the image stay straight!

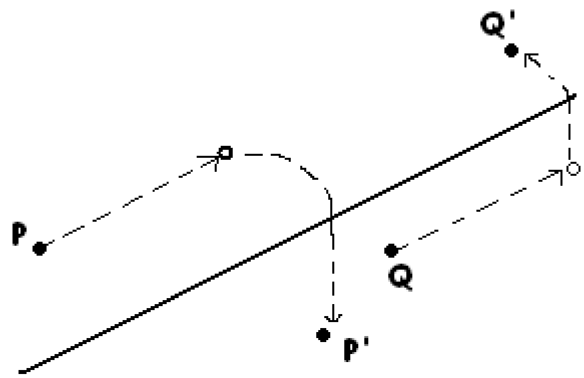
QUESTION: Have the programmers of smart-phone maps been clever and written a subroutine that straightens all the bent roads as you perform the dilation? Or is it just a mathematical property of dilations that they, somehow, are sure to keep straight lines straight?

We'll reveal all later on!

Carrying on with basic examples …

EXAMPLE 4: Some people like to give a special name to a special combination of a translation and a reflection.

A translation followed by a reflection in a line parallel to the direction of the translation is called a glide reflection.

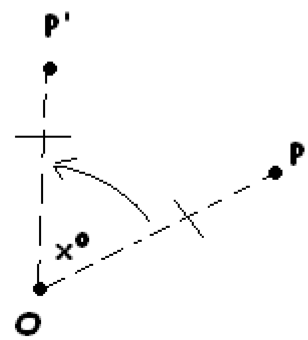


This appeals to our humanness: we see glide reflections whenever we walk on the beach.

EXAMPLE 5: A rotation about a point O through an angle x^o takes a point P, different from O, to a new point P' with the properties:

i) $OP = OP'$
ii) $m\angle POP' = x^o$

The rotation leaves the point O itself fixed in place.



It is assumed that angles are measured in a counter-clockwise direction.

A rotation of 180^o is called a half turn and a rotation of 360^o is called a full turn.
(A full turn brings all points to their original positions.)

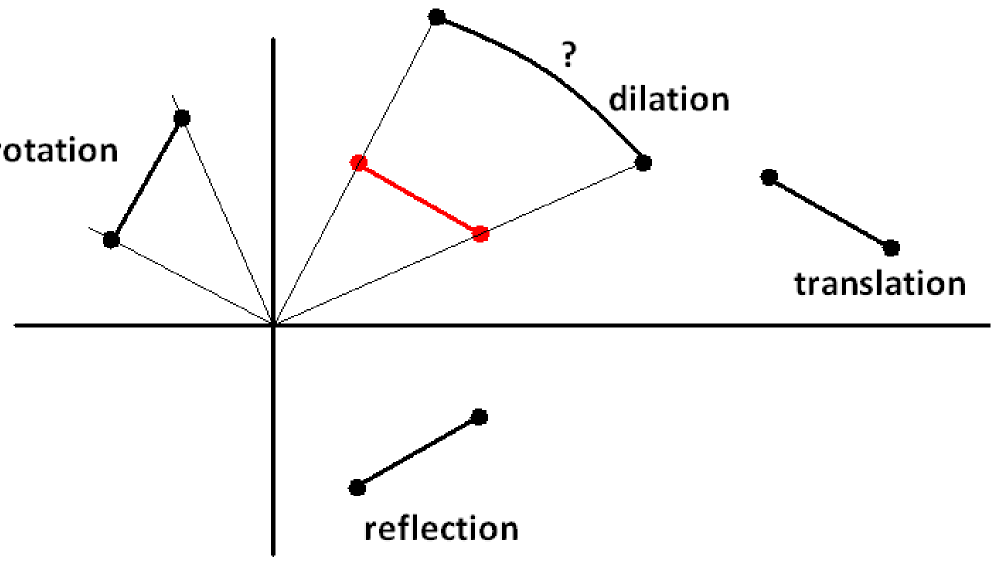
Question: Describe a rotation of 0°. (It isn't much a mapping at all!)

ISOMETRIES

The four mappings describe here – translation, reflection , rotation, and dilation - along with the fifth special example of a glide reflection, are the standard examples of mappings discussed in the school curriculum. But mappings can be wild and crazy and different types of mappings do arise in different branches of mathematics.

But in this discussion we'll focus our attention on these five classical examples.

Here is a picture of a line segment (in red) and its images under each of the basic transformations:



Apart from the query of whether or not dilations map straight line segments to straight segments, it is clear that dilations are qualitatively different from translations, reflections, and rotations:

Dilations change distances between points.

Question: This statement is not quite true. For what scale factor k does a dilation preserve distances between points?

Definition: A mapping is called an *isometry* if it keeps distances between points unchanged.

(More precisely, a mapping is an isometry if, for any two points P and Q in the plane, with images P' and Q', respectively, we are sure to have $PQ = P'Q'$.)

Comment: The prefix *iso-* comes from the Greek for "the same" and *metry* comes from the word *metria* for "measure."

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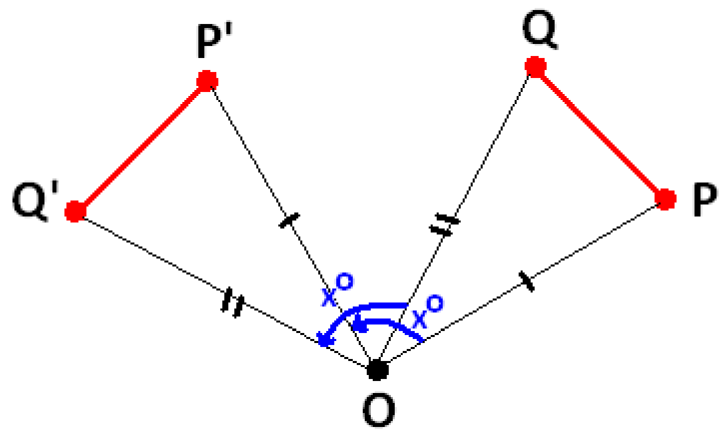
It seems that:

TRANSLATIONS, REFLECTIONS and ROTATIONS ARE ISOMETRIES.

Let's prove this claim for rotations:

Suppose we have a rotation of x° about a point O that takes point P to P' and point Q to Q'.



By the definition of a rotation we have: $OP = OP'$ and $OQ = OQ'$.

Also, if $\angle P'OQ = a$, then

$$\angle QOP = x - a$$
$$\angle Q'OP' = x - a$$

and so $\angle QOP \cong \angle Q'OP'$.

By the SAS principle it follows that $\Delta QOP \cong \Delta Q'OP'$. Thus $PQ = P'Q'$ and the rotation has preserved distances between points.

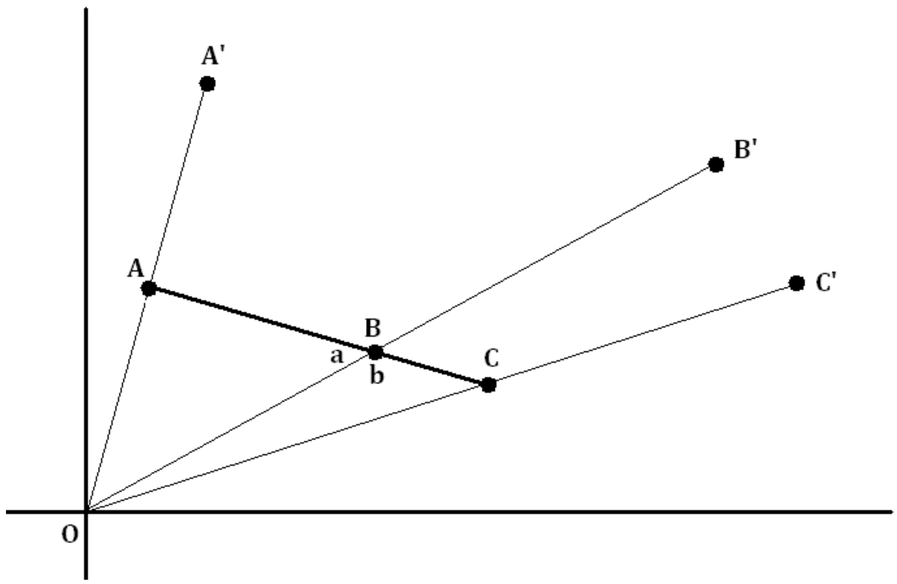
Exercise 3: Prove that translations are isometries. (HINT: Use properties of parallelograms, or draw a diagonal line and look for congruent triangles.)

Exercise 4: Prove that reflections are isometries. (Watch out! There are several cases to consider for the location of the each initial point under consideration: on one side, on the other side, or on the line of reflection.)

SOLVING THE SMART-PHONE MAP MYSTERY

Theorem: *A dilation maps three collinear points to three collinear positions.*

Proof: Assume the dilation is centered about the origin O and has scale factor k. Suppose the three collinear points A, B and C are mapped to positions A', B', and C', respectively. Our goal is to prove that A', B', and C' are collinear. Mark the angles a and b as shown. We have $a+b=180^\circ$.



Now $OA'=k\cdot OA$, $OB'=k\cdot OB$, and $OC'=k\cdot OC$. It follows that $\Delta A'OB' \sim \Delta AOB$ by the SAS principle. (Sides come in the same ratio k and the triangles share a common angle at O.) Thus $\angle A'B'O=a$ as they are matching angles in similar triangles.

Similarly $\Delta B'OC' \sim \Delta BOC$ and $\angle OB'C'=b$.

Thus $\angle A'B'C'=\angle A'B'O+\angle OB'C'=a+b=180^\circ$, which shows that A', B', and C' are indeed collinear.

Comment: One can show, in general, that a dilation maps any set of collinear points to collinear positions. In particular, dilations take straight line segments to straight line segments. Thus the effect of preserving the straightness of roads with the two-finger swipe on smart phone maps is a consequence of geometry. The programmers added no alterations to accommodate!

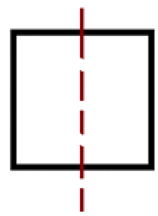
FINALLY … WHAT IS SYMMETRY?

We are now ready to say what a symmetry is!

> *A figure in the plane is said to possess* symmetry *if there is a mapping that maps the figure onto itself.*

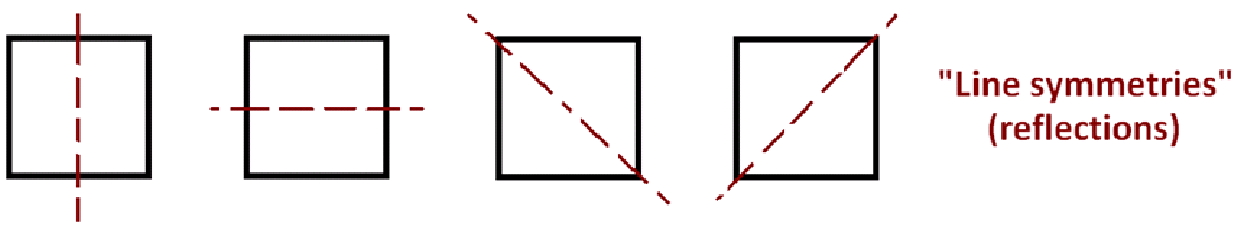
(This definition is still somewhat loose. We expect every point in the figure to be mapped to a point in the figure and no points are to be missed in the mapping, that is, that every point in the figure is the image of some point. We also still haven't given a precise definition of what a mapping really is. Hmm!)

For example, we would all agree that a square has "vertical line symmetry."

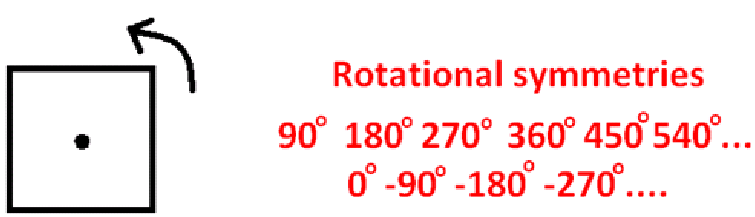


By this, we mean that the refection about this line maps the square onto itself.

The square, in fact, has four line symmetries.

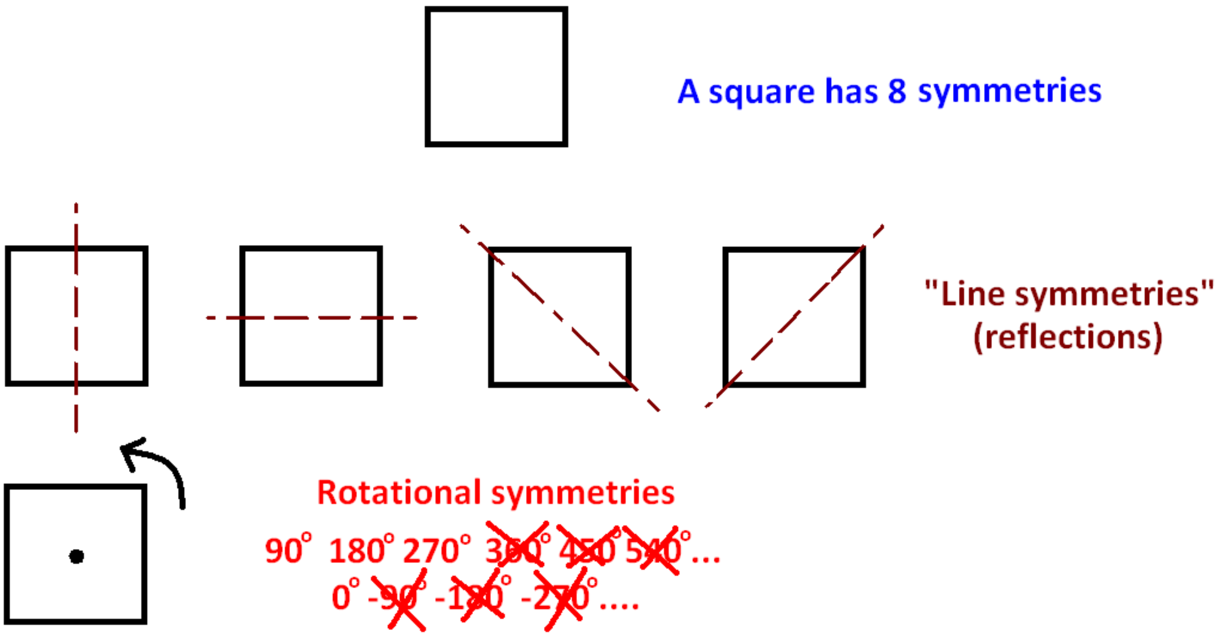


The square also possesses an infinite number of rotational symmetries:



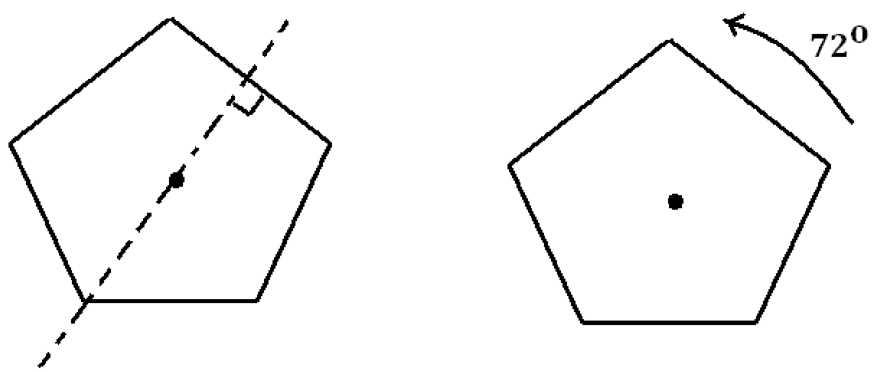
… though most would say that this list boils down to basically three essentially distinct types of rotations, or four if you are willing to consider the trivial rotation of 0°. (For the sake of completeness, mathematicians do include this trivial rotation in the list.)

All in all we say that a square has EIGHT symmetries:



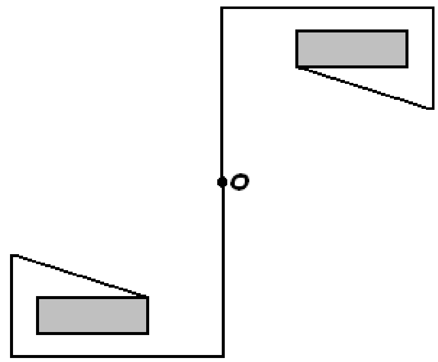
EXAMPLE: *Identify all the symmetries of a regular pentagon. How many line symmetries does it possess? How many rotational symmetries?*

> **Solution:** A regular pentagon has five line symmetries (reflection symmetries), one about each line through the center of the figure and a vertex. It has five rotational symmetries, rotations of $0^\circ, 72^\circ, 144^\circ, 216^\circ, 288^\circ$ about the center (which includes the trivial rotation.) .



Exercise 5: Identify all the symmetries of a regular octagon. How many line symmetries does it possess? How many rotational symmetries?

Jargon: The following figure, for example, has 180^o rotational symmetry about the point O.



Some like to call 180° rotational symmetry about a point a point symmetry.

Exercise 6: Which regular polygons possess point symmetry?

Exercise 7: If possible draw an example of each of the following:
i) A quadrilateral with exactly four lines of reflection symmetry.
ii) A quadrilateral with exactly two lines of reflection symmetry.
iii) A quadrilateral with exactly one line of reflection symmetry.

Exercise 8:
a) List all the (capital) letters of the alphabet that have horizontal line symmetry.
b) List all the (capital) letters of the alphabet that have vertical line symmetry.
c) DICED is a word with horizontal symmetry. Can you think of another word (when written with capital letters) with this property? Can you think of a word with vertical symmetry?

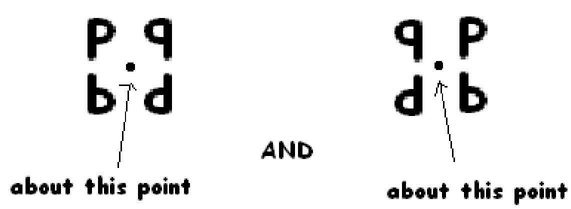
EXAMPLE: Here's a design with a repeating motif that extends infinitely far to the left and to the right:

Such a design is called a <u>frieze pattern</u>.

This design has:

- **TRANSLATIONAL SYMMETRY**
  One can shift this pattern to the left or to the right to have the design overlap upon itself.

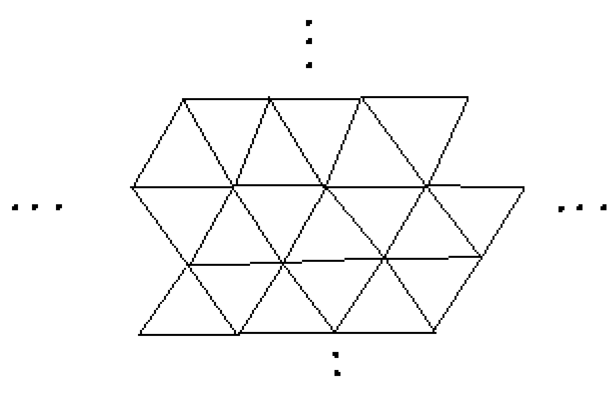
- **TWO DIFFERENT TYPES OF POINT SYMMETRY**
  Namely:



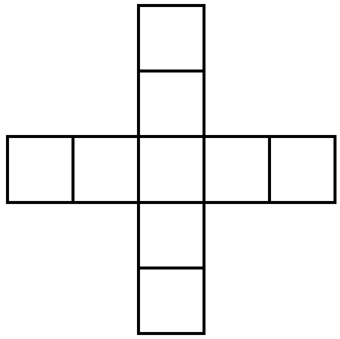
- **REFLECTION SYMMETRY**
  Both vertical and horizontal

- **GLIDE REFLECTION SYMMETRY**
  One can translate and then reflect about a horizontal line.

Exercise 9: Identify all the symmetries possessed by an infinite array of equilateral triangles that extend across the entire plane.



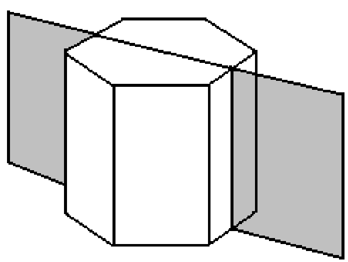
Exercise 10: List all the symmetries of a design of nine squares arranged as follows:



COMMENT: The concept of "symmetry" extends to three-dimensional figures as well.

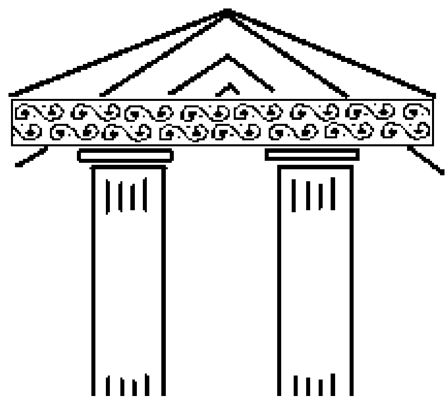
For example, a "reflection in a plane" takes a point on one side of a plane to a point on the opposite side so that the line segment connecting the two points is perpendicular to the plane and bisected by it. (And all points on the plane are held fixed in place by the reflection.)

We can say that a regular hexagonal cylinder, for example, has "plane symmetry."



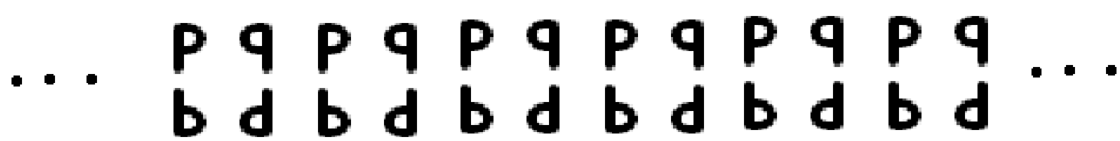
A SPECTACULAR AND SURPRISING APPLICATION OF SYMMETRY: FRIEZE PATTERNS

In classical architecture a frieze is a horizontal structure, usually imprinted with decoration, resting on top of columns. The modern equivalent is a horizontal strip of wallpaper used to decorate the top portion of a wall just below the ceiling.



Usually the design on a frieze is composed of repeated copies of a single motif. If we assume that the frieze extends infinitely far in both directions, then the design possesses translational symmetry: shift the design the appropriate distance to the left and the design looks identical

.



By a <u>frieze pattern</u> we mean an infinitely long strip imprinted with a design given by a repeating motif.

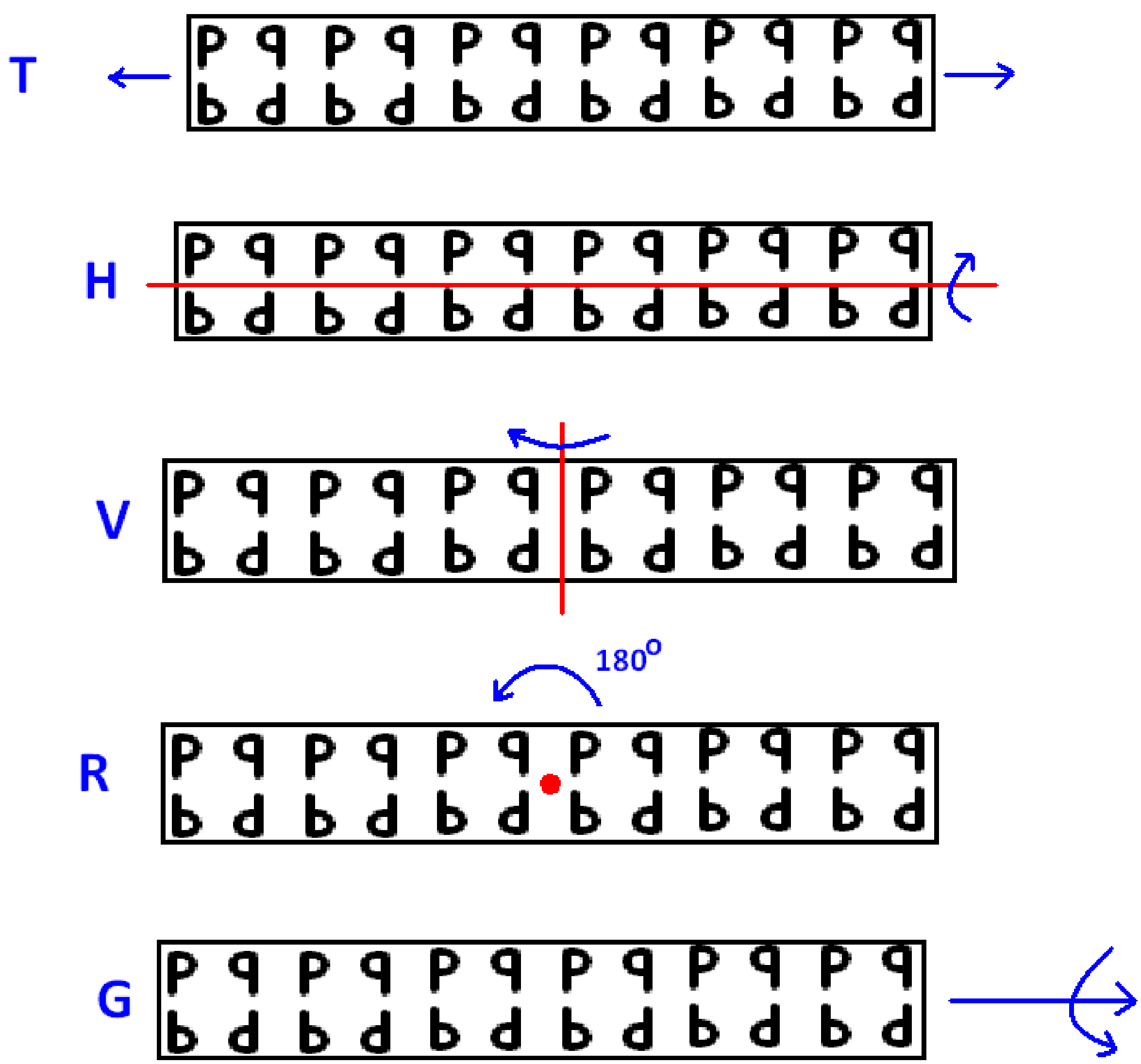
Mathematicians are interested in the symmetry properties of frieze patterns.

Each frieze pattern, by definition, is symmetrical under a translation T in the direction of the strip. A frieze pattern might also be symmetrical about a horizontal reflection (H), a vertical reflection (V), a rotation of 180° about some point in the design (R), a glide reflection (G), or some collection of these five basic transformations. The frieze pattern above with the letters p, b, q, d above possesses all five symmetries.

ACTIVITY: REALLY CHECK THIS!

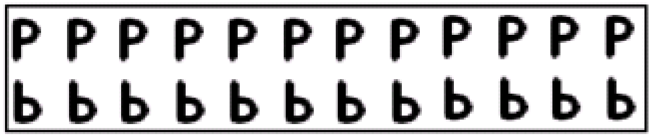
Make a cardboard copy of this design with the same pattern copied on each side. (Pretend the cardboard is transparent so that the design you draw on the back of the card is the "see through" version of the design on the front.)

Verify that the card looks the same if you perform each of the basic transformations on it:



Not all frieze patterns possess all these symmetries.

ACTIVITY: Make a cardboard copy of the following frieze design with the pattern drawn on both sides (as though the card were transparent).



Verify that this design possesses symmetries T, H, and G, but not V nor R.

By definition, every frieze design possesses T as a symmetry. This leaves 16 possible combinations for the types of symmetries that a frieze pattern might possess.

T	**TV**	**TVH**	**THG** ✓	**TVHG**	**TVHGR** ✓
	TH	**TVR**	**THR**	**TVHR**	
	TG	**TVG**	**TGR**	**TVGR**	
	TR			**THGR**	

We've already seen examples of THG and $TVHGR$. Can all other combinations of symmetries actually occur in a frieze pattern?

Notice that a frieze pattern cannot possess the symmetries T and H without G as well. (A translation followed by a horizontal reflection is a glide reflection.) So we'll never have a frieze design with the symmetries TH, TVH, THR, or $TVHR$. That leaves 12 possible symmetry combinations.

T	**TV**	~~**TVH**~~	**THG** ✓	**TVHG**	**TVHGR** ✓
	~~**TH**~~	**TVR**	~~**THR**~~	~~**TVHR**~~	
	TG	**TVG**	**TGR**	**TVGR**	
	TR			**THGR**	

ACTIVITY: Make a card with the word MATHEMATICS! on one side and a copy of this word on the back (but written as though the card were transparent and you were seeing the letters though the other side of the card. It won't look like the word MATHEMATICS! on this side.)

Using your card …

> Verify that the action of a V and an R is equivalent to an action of a G. This means that any frieze design that possesses V and R must possess G as well.
>
> Verify that the action of a V and an G is equivalent to an action of a R. This means that any frieze design that possesses V and G must possess R as well.
>
> Verify that the action of a G and an R is equivalent to an action of a V. This means that any frieze design that possesses G and R must possess V as well.

Show that these observations leave only SEVEN possible symmetry combinations for frieze patterns.

It turns out that frieze patterns with each of these seven symmetry combinations do exists.

ACTIVITY:

> Design a frieze pattern that possess only T for its symmetry.
>
> Design a frieze pattern that possess only T and V for its symmetry.
>
> Design a frieze pattern that possess only T and G for its symmetry.
>
> Design a frieze pattern that possess only T and R for its symmetry.
>
> Design a frieze pattern that possess only T, V, R and G for its symmetry.

We've already seen a design with T, V, G, H, and R for its symmetry; and we've already seen a design with just T, H, and G for its symmetry.

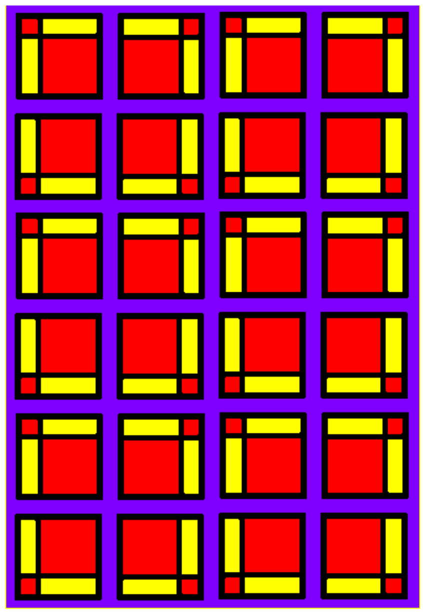
I'll show you my designs for each of these symmetry combinations two pages from now.

So we have shown:

There are only seven symmetry types of frieze patterns.

Of course, we could extend these considerations up a dimension and consider the symmetries of two-dimensional wallpaper designs (assumed to extend infinitely far in all directions).

For example, the design below as translational symmetry (in multiple directions), 90° rotational symmetry, reflection symmetries, and so on.



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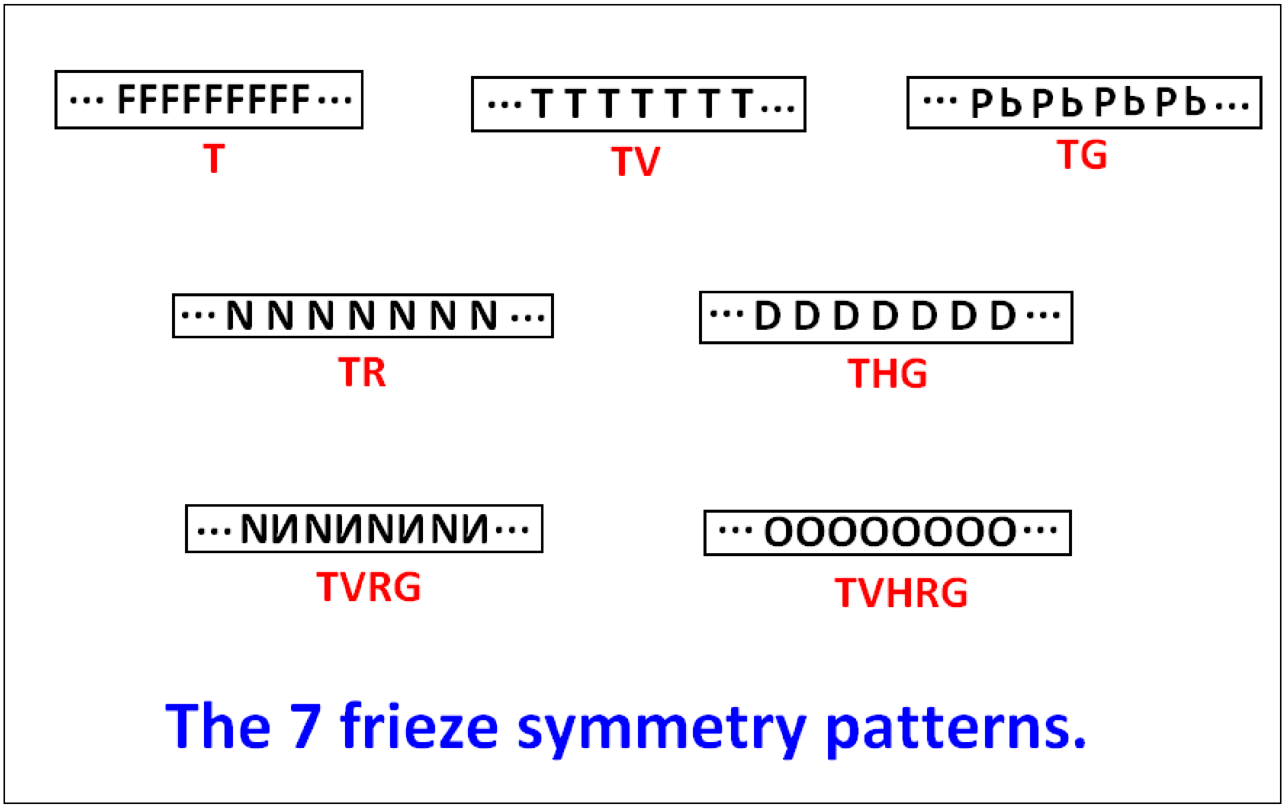
Mathematicians have proved that are 17 different wallpaper symmetry patterns!

As promised …


… FFFFFFFFF …
T
… T T T T T T T …
TV
… PЬPЬPЬPЬ …
TG
… N N N N N N N …
TR
… D D D D D D D …
THG
… NИNИNИNИ …
TVRG
… OOOOOOOO …
TVHRG
The 7 frieze symmetry patterns.

THE MATHEMATICS OF FRACTALS

Let's now talk about a very modern field of geometry: fractal geometry.

This subject dovetails our thinking on mappings and symmetry. We say a figure is symmetrical if there is a mapping the maps the entire figure back onto itself. A fractal, on the other hand, is a figure composed of sections that are the result of mapping the entire figure onto parts of itself. In fact, each section is an exact scaled copy of the entire original shape. (This is vague and confusing, I know!)

Our discussions on symmetries made no use of dilations. Fractals make significant use of dilations and the change of scale they represent.

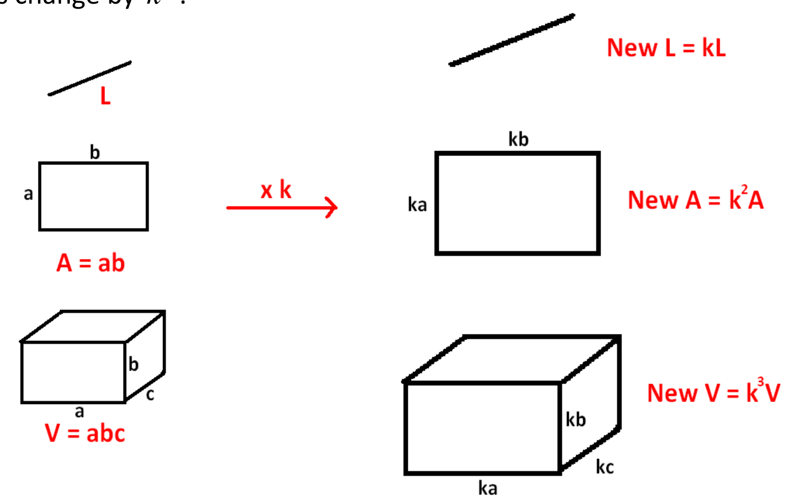
You no doubt have heard word fractal before and have probably seen pictures of these infinitely jaggedly things people call fractals. In this lecture we'll talk about what fractals are, develop a definition for them, and get going on some of their mathematics.

The word *fractal* was coined in 1980 by Belgian mathematician Benoit Mandelbrot (1924 – 2010). Mandelbrot chose the name "fractal" because it reminds us of the word "fraction." He did this for good reason. Mandelbrot realised that these self-similar shapes have the property of not being one-dimensional, or two-dimensional, or even three-dimensional,
but are instead of fractional dimension. (Now we have absolutely no clue what I am talking about!)

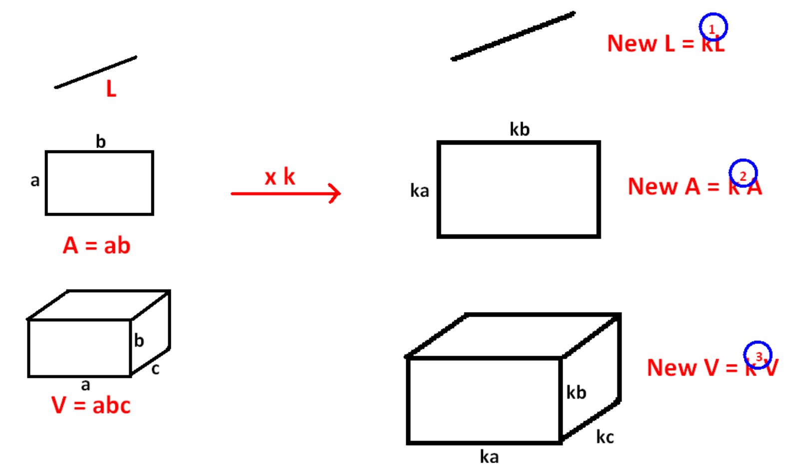
To explain, we first need to describe what we mean by dimension.

SCALE AND DIMENSION

Recall from a study of scale, that if we enlarge a figure by a scale factor k (or decrease the image if k happens to be smaller than 1), then all lengths in that picture change by a factor k, all areas change by k^2, and all volumes change by k^3.



If you like, we can say that all lengths are changing by a factor of k^1.



This gives us a way to <u>define</u> the dimension of a shape in geometry.

> If an object scales as k^2, then it is two-dimensional, an area.
> If an object scales as k^1, then it is one-dimensional, a length.
> If an object scales as k^3, then it is three-dimensional, a volume.

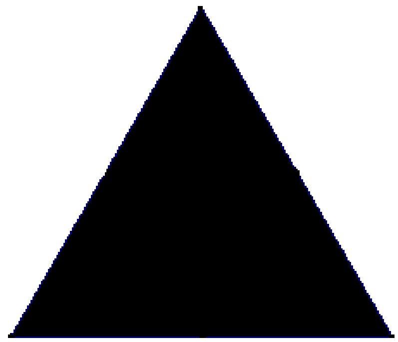
Wouldn't it be surprising to find some figures that scale in-between these numbers, say as $k^{1.7}$, something between a length and an area, or as $k^{2.4}$, between an area and a volume?

Let's find such shapes!

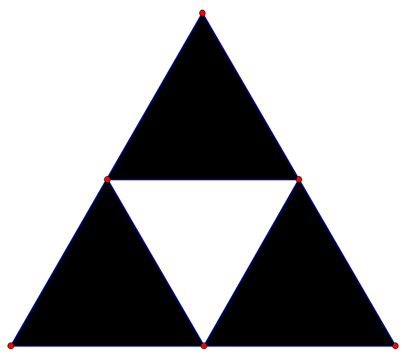
THE CLASSIC EXAMPLE: SIERPINKI'S TRIANGLE

In 1916, Polish mathematician Waclaw Sierpinski examined the following construction:

Begin with a solid equilateral triangle.

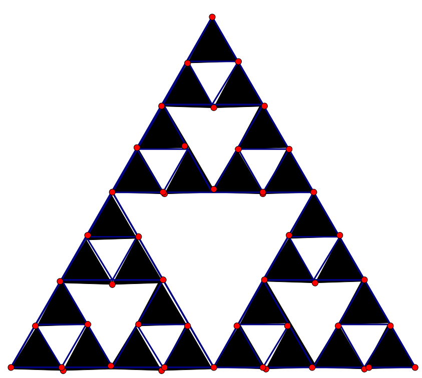


Imagine it divided into four congruent equilateral triangles and remove its center one.



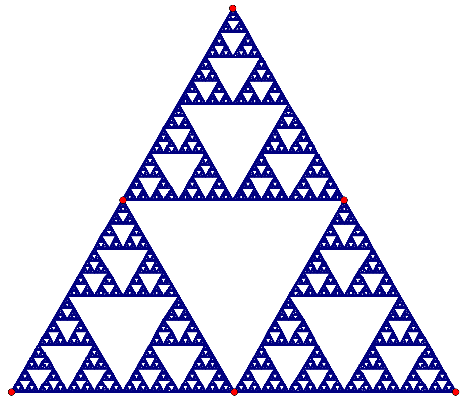
The total area of this figure is $\frac{3}{4}$ the original area.

Now remove the middle quarters of each of the three solid triangles we still see.



Question: Explain why the total area of the solid triangles in this diagram is $\frac{9}{16}$ of the area of the original triangle.

Now remove the middle sections of the each of the solid triangles in this diagram, and repeat the process over and over and over and over again.



At each stage the area of the figure decreases by a factor of $\frac{3}{4}$. So if the original triangle has area 1 square unit, the areas of the figures at each stage are:

$$\begin{aligned}
&1,\\
&\frac{3}{4} = 0.75,\\
&\frac{3}{4} \times 0.75 \approx 0.56,\\
&\frac{3}{4} \times 0.56 \approx 0.42,\\
&\frac{3}{4} \times 0.42 \approx 0.31\\
&\frac{3}{4} \times 0.31 \approx 0.24\\
&\frac{3}{4} \times 0.24 \approx 0.18\\
&\vdots
\end{aligned}$$

Are these values decrease to zero? If I could continue this process forever, would the resulting figure "contain no area" and so be just be a nest of lengths, the boundaries of all the regons?

Or does the some area "survive" and the figure stays an area since it starts that way and stays that way at every stage?

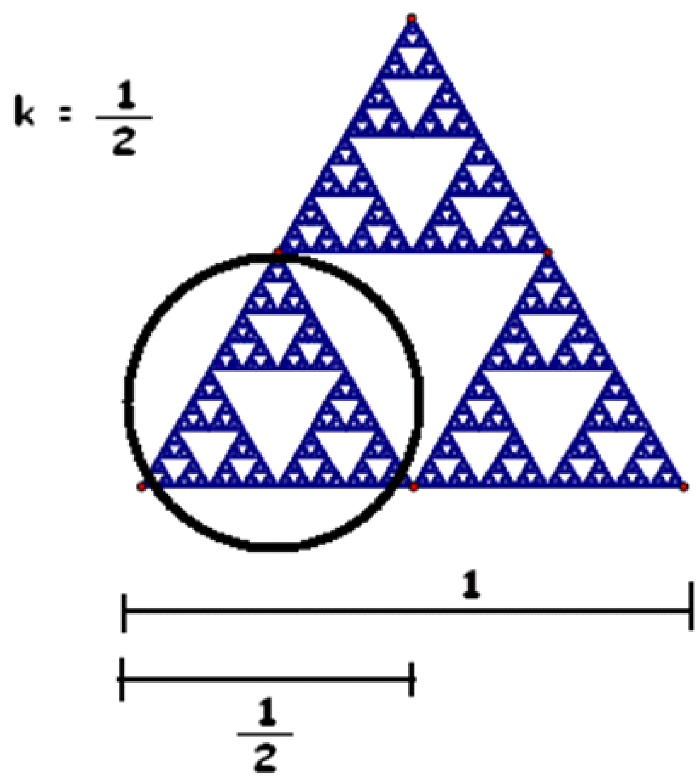
Sierpinsky's triangle is the figure that would result if we could carry this process of removing middle triangles forever. We as humans can never conduct an infinite process and so we will never see a true representation of Sierpinsky's triangle. But we do feel we can image one in our minds.

So the question at hand is …

> **IS THE SIERPINSKY'S TRIANGLE, the true result we would have at the end of the infinite process, A LENGTH OR AN AREA?**

To answer this need to examine how the figure behaves in scaling by a factor k. If its size changes by a factor k^2, then we know it is an area. If its size changes by k^1, then we know it is length.

But notice first that Sierpinski's triangle is composed of three pieces, each an exact scaled copy of the entire shape. In fact, we can see that the scale factor for each piece is $k = \frac{1}{2}$, as each small piece has base half the length of the base of the whole figure.



Also , since the whole figure is composed of three identical pieces, we have that the "size" of any one piece is one-third the size of the whole thing. (If Sierpinsky's triangle is an area, then I would say that the area of any one piece is one-third the area of the whole triangle. If Sierpinsky's triangle is a length, then I would say that the total length of any one piece is one-third the total length of the whole triangle.)

Alright … IS SIERPINSKI'S TRIANGLE AN AREA?

If it is, then each piece scales as k^2. We get a contradiction:

If Siepinsky's Triangle is an area ...

By inspection:

$$\boxed{\text{Area one piece} = \frac{1}{3} \cdot \text{Whole area}}$$

By scaling:

$$\boxed{\text{Area one piece} = k^2 \cdot \text{Whole area}}$$

$$k^2 \cdot \text{whole area} = \frac{1}{3} \cdot \text{whole area}$$

$$k^2 = \frac{1}{3}$$

$$\left(\frac{1}{2}\right)^2 = \frac{1}{3}$$

Hmm…. IS SIERPINSKI'S TRIANGLE A LENGTH?

If Siepinsky's Triangle is a length ...

By inspection:

$$\boxed{\text{Total length one piece} = \frac{1}{3} \cdot \text{Whole length}}$$

By scaling:

$$\boxed{\text{Total length one piece} = k \cdot \text{Whole length}}$$

$$k \cdot \text{whole length} = \frac{1}{3} \cdot \text{whole length}$$

$$k = \frac{1}{3}$$

Another contradiction!

Alright, let's just say it is d -dimensional and let the mathematics tell me what d should be.

By inspection:

Total Size of one piece = $\frac{1}{3}\cdot$Whole size

By scaling:

Total Size of one piece = $k^d\cdot$ Whole size

$$k^d \cdot \text{whole size} = \frac{1}{3} \cdot \text{whole size}$$

$$k^d = \frac{1}{3}$$

$$\left(\frac{1}{2}\right)^d = \frac{1}{3}$$

$$\boxed{2^d = 3}$$

We need a value d so that $2^d = 3$.

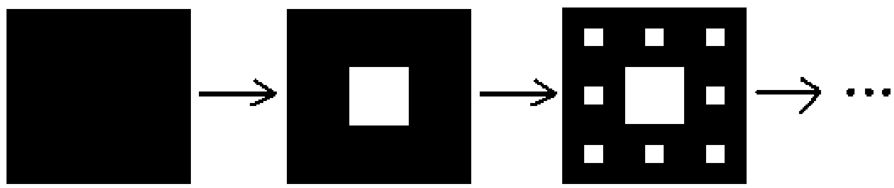
$d = 2$ is too big.
$d = 1$ is too small.
$d = 1.5$, on a calculator, is too small.
$d = 1.6$, is just a tad too big.

It turns out that $d = 1.58....$ works.

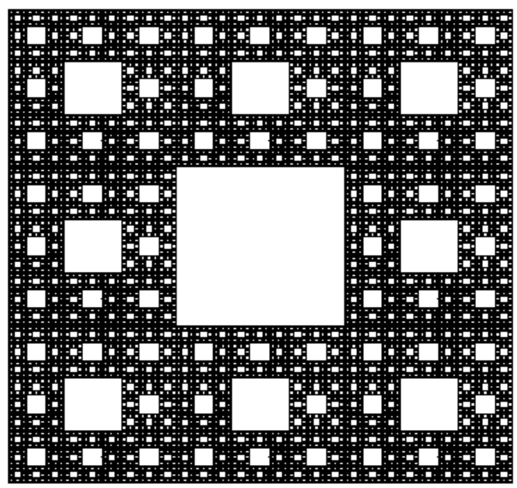
Sierpinski's triangle is an object somewhere between being a length and an area (slightly more area-like than length-like!)

SIERPINSKY'S MAT

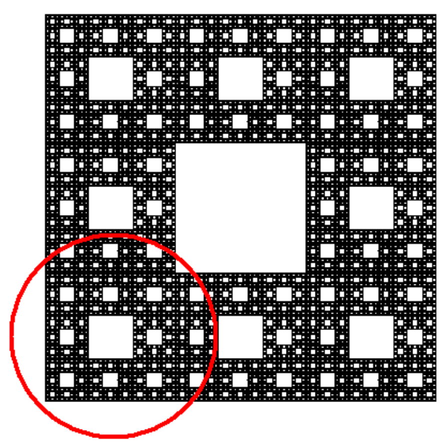
This time start with a square, divide it into nine congruent sub-squares and remove the innermost square. Repeat this process on eight remaining squares, and then again on all the squares that survive, and again, and again, ad infinitum.



The "end result," if we could actually see it, is Sierpinsky's mat:



The mat is composed of EIGHT copies of itself each at one-third the scale.



Eight copies.

Scale: $k = \frac{1}{3}$

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The same mathematics show that it has a dimension d that satisfies $3^d = 8$. We get that $d = 1.89\ldots$.

By inspection:

$\boxed{\text{Total Size of one piece} = \frac{1}{8} \cdot \text{Whole size}}$

By scaling:

$\boxed{\text{Total Size of one piece} = k^{d} \cdot \text{Whole size}}$

$$k^d \cdot \text{whole size} = \frac{1}{8} \cdot \text{whole size}$$

$$\left(\frac{1}{3}\right)^d = \frac{1}{8}$$

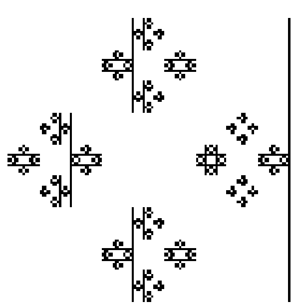
$$3^d = 8$$

We can now see a definition for a fractal.

> *A* <u>fractal</u> *is a figure composed of pieces that are each themselves a scaled copy of the entire original figure.*
>
> *Figures with such self-similarity can have dimensions that are not integers.*

I need to point out again that these are very much a construction of the mind. Fractals are the end result of an <u>infinite</u> process. We can never fully construct these objects in our human lifetimes.

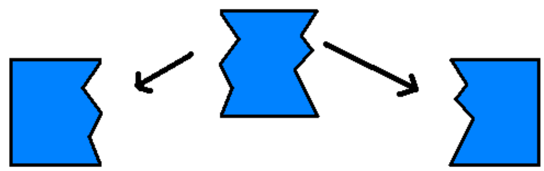
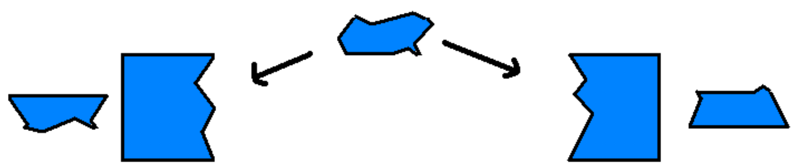
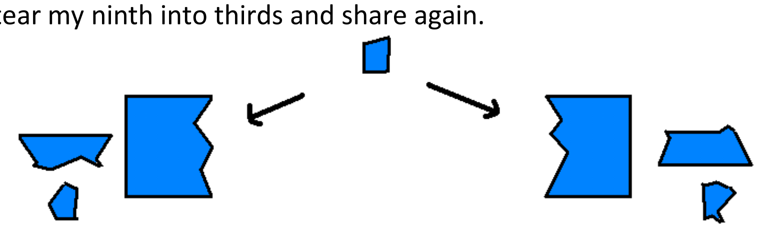
Exercise 11: What is the fractal dimension of this fractal?



Exercise 12. The Seirpinski sponge is constructed from a cube by dividing it into 27 small cubes and removing the very center cube and the center cube of each face. The action is then repeated on the 20 remaining cubes, and then repeated over and over again in the cubes that remain after each iteration. What is the fractal dimension of the Sierpinski sponge?

ANOTHER INFINITE PROCESS:
PAPER TEARING and the GEOMETRIC SERIES FORMULA

There is an infinite process that most people feel they <u>can</u> visualize and understand the "final" result. Here Albert explains a paper-tearing exercise to his two friends Bilbert and Cuthbert:

> Here's a piece of paper which I am tearing into thirds. I'll give each of you a third and keep the remaining third for myself.
>
> 
>
> Actually, I am a generous guy, and so I am going to tear my third into thirds again and give you each a piece.
>
> 
>
> Right now you each have a third plus a third of a third ($1/3+1/9$) and I am holding onto a small piece one-ninth in size.
>
> Actually let me tear my ninth into thirds and share again.
>
> 
>
> Now you each have $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$ of paper and I have a tiny amount – which I'll share again by tearing into thirds.
>
> Now imagine I do this forever. The amount of paper I possess dwindles away to nothing and the paper is being equally shared among you both.
>
> Now ask: In the end, how much paper will you each get?
>
> The math says that you will each receive $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$ paper. But logic says that since <u>all</u> the paper is shared equally between you both, you each get half the paper. Thus:
>
> $$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{2}.$$

In the same way if I share a piece of paper with 19 friends by tearing it into 20 pieces and divvying up my share over and over again we'll see:

$$\frac{1}{20}+\left(\frac{1}{20}\right)^2+\left(\frac{1}{20}\right)^3+\cdots=\frac{1}{19}$$

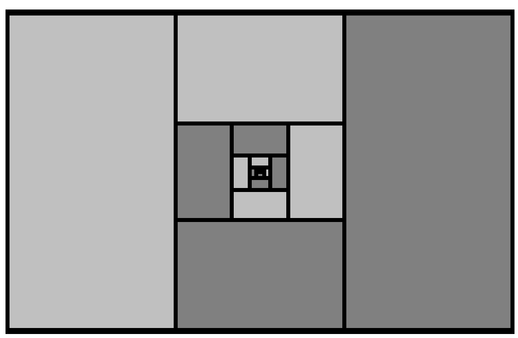
In general we have the formula:

$$\frac{1}{N}+\left(\frac{1}{N}\right)^2+\left(\frac{1}{N}\right)^3+\cdots=\frac{1}{N-1}$$

This final formula is a version of the geometric series formula one learns in a pre-calculus class.

The point is we can get to this infinite formula in a way that makes intuitive sense: the paper Albert holds really will dwindle to nothing.

Exercise 13: In what way does the following diagram illustrate the geometric series formula for $N=3$?

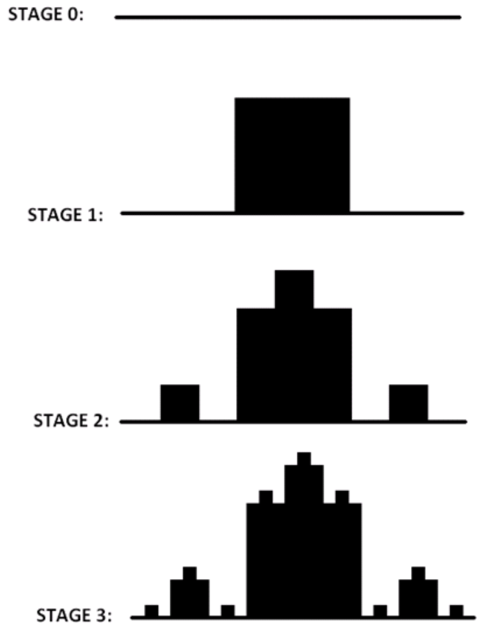


Optional Challenge: Care to find a picture that shows $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots=\frac{1}{3}$?

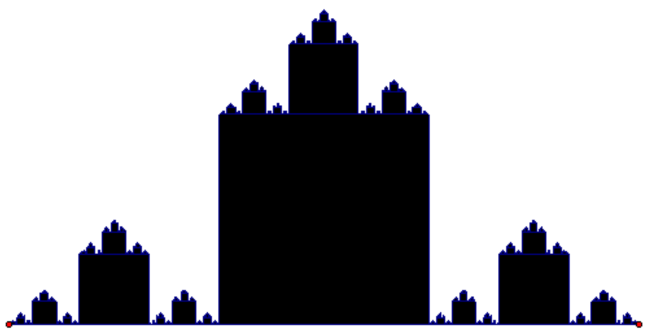
Exercise 14: Use the geometric formula to show that the infinitely long decimal $0.1111111...$ equals the fraction $\frac{1}{9}$.

Exercise 15: Consider the following "city-scape" construction.

Begin with horizontal segment and place on its middle-third a solid square. Repeat this process for each horizontal section that appears, ad infinitum.

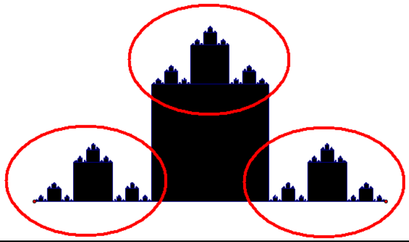


The "end result" is a indeed reminiscent of a city-scape.



Prove that this design is no more than $\frac{1}{2}$ a unit high.

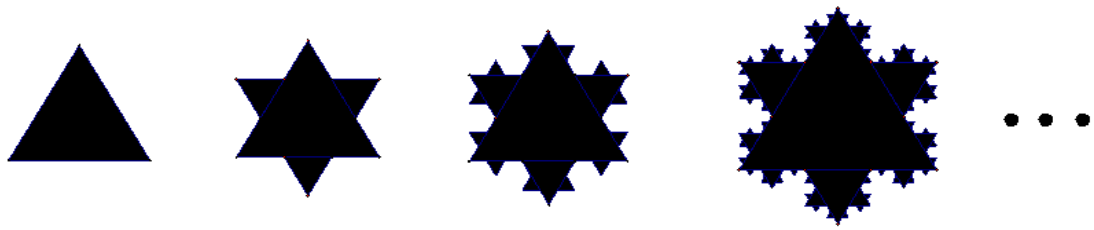
Comment: This figure is <u>not</u> a fractal. It is true that the figure is composed of three pieces that are scaled copies of the entire figure, but there is a fourth piece – the central solid square – that is not a copy of the entire figure. (A fractal is composed solely of scaled copies of the original.)



ANOTHER CLASSIC EXAMPLE: THE KOCH SNOWFLAKE

The following construction was studied by Helge von Koch in 1904.

Start with an equilateral triangle and add an equilateral triangle to the middle third of each boundary segment. Repeat, ad infinitum.



The resulting figure is called the Koch Snowflake.

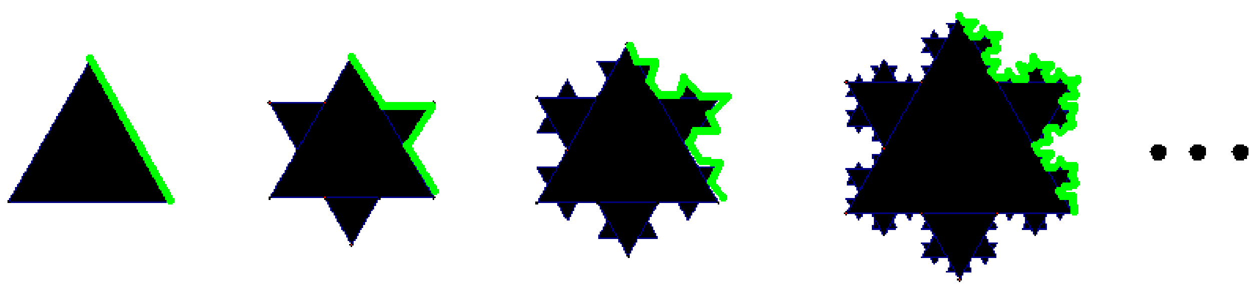
Exercise 16:

a) Show that the perimeter of the figures in this construction increase by a factor of $\frac{4}{3}$ from one stage to the next. What is the perimeter of the Koch snowflake?

b) If the area of the initial equilateral triangle is 1 square unit, prove that the area of the Koch snowflake is 1.6 square units, exactly!

The Koch snowflake is an example of a geometric figure of infinite perimeter enclosing a finite area!

Exercise 17: The Koch snowflake is not a fractal, but the boundary of one side of the snowflake is.



The section of boundary shown in green here is composed of four copies of itself, each copy at one-third the scale.

What is the fractal dimension d of this section of boundary?

FRACTALS AND NATURAL PHENOMENA

The Koch snowflake has the property that it has an extraordinarily large perimeter (in fact, infinite perimeter) for a shape of finite area. A three-dimensional analog would be a shape of fixed volume with a very large surface area.

Natural biological systems often need to utilize such scenarios to garner efficient function. For example, the small intestine of the human body absorbs nutrients through its lining. The greater the surface area exposed to food sources, the more nutrients it can absorb. But the human body is a fixed size and there are limitations on how much volume the small intestine can occupy.

To maximize surface area for a small volume, the inner surface of the intestine is not flat, but rippled. Moreover, the ripples of tissue are covered with villi, small projections of tissue that increase surface area. And on these villi are microvilli, increasing the surface area even more.

A typical small intestine is about 6 meters long yet the surface area it offers is 250 square meters – the size of a tennis court!

The typical adult lung offers a surface area of about 100 square meters. (Why is large surface area for lungs necessary with regard to their function?)

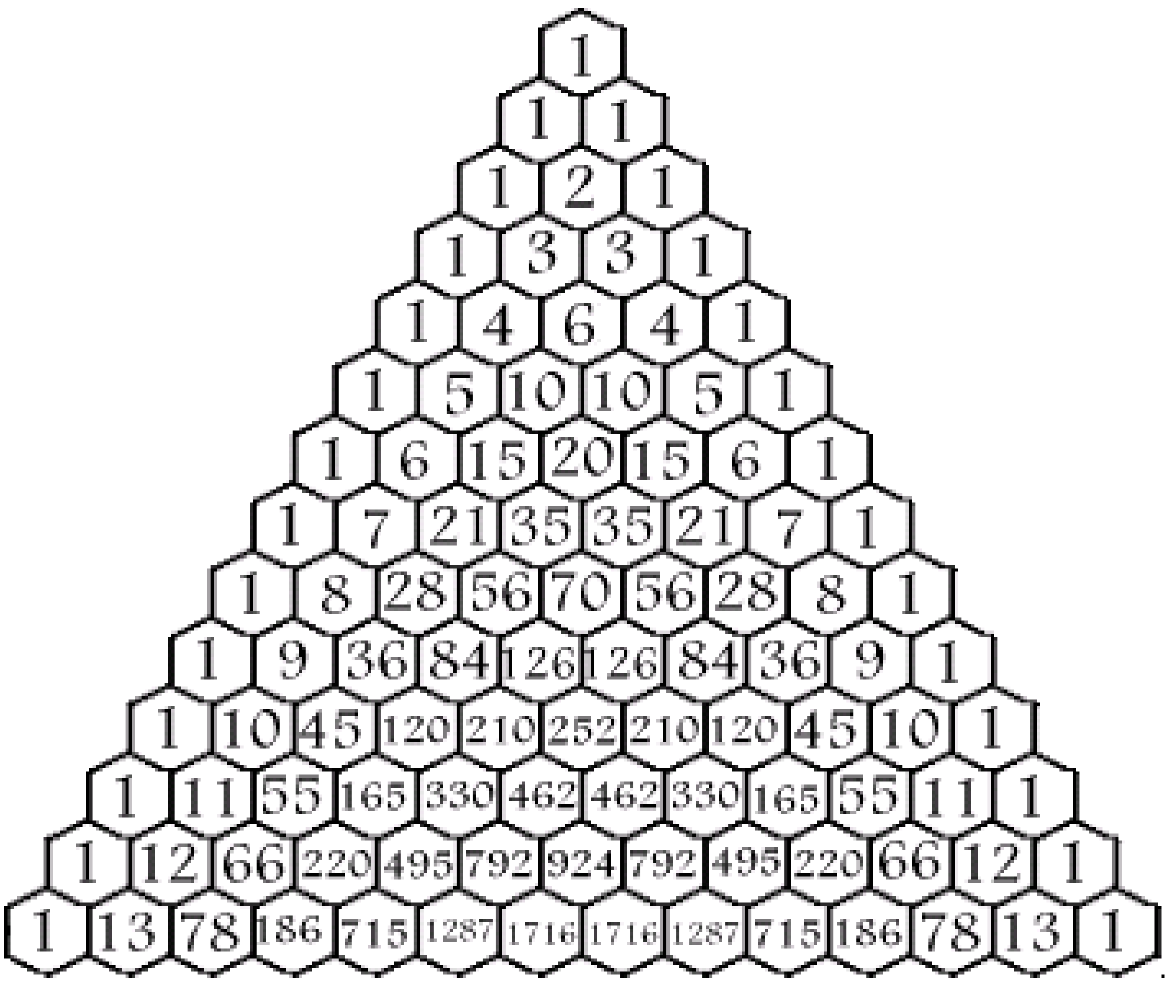
In the marine world, a sea sponge is a colony of small animals that garners its food from contact with the surrounding water. In order to increase surface area, a sea sponge leaves a large number of holes in its structure.

Exercise 18: Why does a block of Swiss cheese dry out faster than a block of cheddar cheese?

Exercise 19: Two walls of identical size are to be painted. One has a smooth surface and the other a textured surface. Which wall will require the greater amount of paint to cover?

A FRACTAL SURPRISE FROM ARITHMETIC

Pascal's triangle possesses the number 1 at its apex and thereafter each number in the triangle is the sum of the two numbers directly above it (with a "blank space" considered the number zero).



Color in all the cells containing on odd number. Do this for the first fifty or so rows of Pascal's triangle.

What do you see?

Can you explain why the pattern you see is sure to continue?