WHY FRACTIONS ARE SO HARD
On how we teach them and What they actually are

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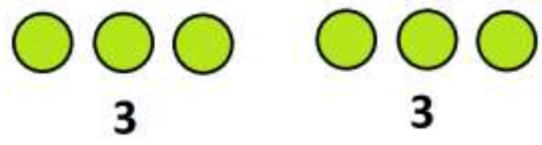
CONTENTS:

MULTIPLICATION

MULTIPLCATON AS GROUPING

In the world of the counting numbers $1, 2, 3, 4$, …. multiplication is usually seen as repeated addition. (Question: Should zero be in this list of counting numbers? Is zero counting anything?)

For example, 2×3 is read as "two groups of three." If our numbers are counting dots, say, then two groups of three looks like this:



and we have six dots.

$$2 \times 3 = 3 + 3 = 6.$$

In the same way:

$4 \times 5 =$ four groups of five $= 5 + 5 + 5 + 5 = 20$,

$1 \times 7 =$ one group of seven $= 7$,

$7 \times 1 =$ seven groups of one $= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$,

and

$635252 \times 98109 =$ lots of groups of 98109.

Comment: Zero is troublesome!

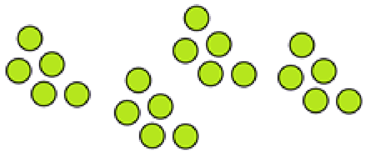
What's 0×5*, no groups of five?*

Is the answer three? If I have three dots, I certainly have no groups of five!

Maybe it is best to leave zero off the list of counting numbers (lest I ask about 0×0, no groups of nothing!).

COMMUTATIVITY IS NOT OBVIOUS!

Here is a picture of four groups of five. There is a total of 20 dots in the picture.



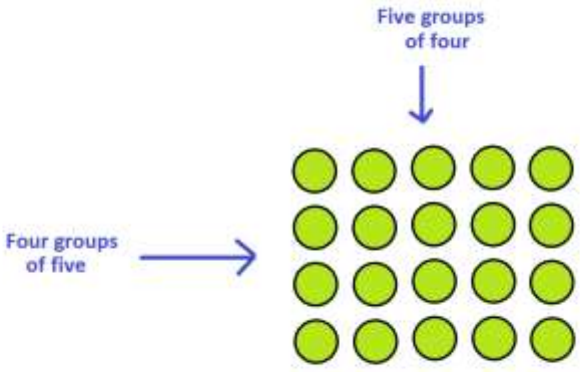
Is it a coincidence that "five groups of four" is also 20 dots?

Let's make it worse:

> *In your mind's eye can you honestly see that a picture of* 635252 *groups of* 98109 *dots will have just as many dots in it as a picture of* 98109 *groups of* 635252 *dots?*

It actually isn't obvious that the products 635252×98109 and 98109×635252 should have the same value. Nor is it philosophically obvious that 4×5 should equal 5×4, unless….

… you draw your picture of four groups of dots in a rectangular array:



Look from the side and you see four groups of five dots as rows, making 20 dots. But change your perspective by 90° and you suddenly see five groups of four as columns. The number of dots in the picture has not changed by your shift of perspective, so five groups of four must, philosophically, give the same answer as four groups of five.

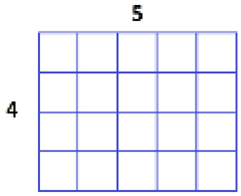
In the same way, by drawing a 17-by-23 array of dots we see that 17×23 must correspond to the same number of dots as 23×17, and even 635252×98109 must match 98109×635252 if we have the patience to draw that large an array of dots.

We conclude that $a \times b = b \times a$, at least for counting numbers a and b .

Comment: In my April 2015 COOL MATH ESSAY (see www.jamestanton.com/?p=1072) the question "Is it obvious that multiplication commutes?" arises in a different context.

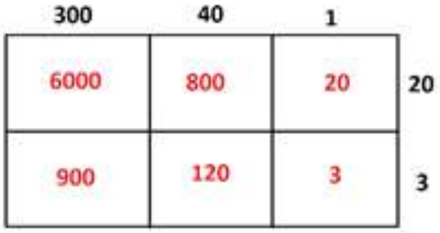
THE AREA MODEL

If we count unit squares instead of dots, we see that the multiplication corresponds to a computation of area. For example, four groups of five units squares make a 4-by-5 rectangle whose area we say is $4\times 5=20$ square units.



4x5 unit squares

This thinking makes computation of large complicated products much more manageable. For example, we can compute 341×23 as $6000+800+900+120+20+3=7843$.



300	40	1	
6000	800	20	20
900	120	3	3

This is, by far, the easiest and most natural way to compute long multiplications. (Now really do read the April 2015 COOL MATH essay at www.jamestanton.com/?p=1072.)

DIVISION

The operation of division is often presented in two different contexts. Surprisingly – and confusingly – they seem philosophically inequivalent, despite computations always seeming to match.

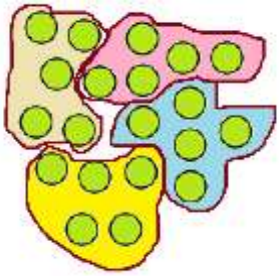
DIVISION AS FINDING GROUPS

As "reverse multiplication," division counts how many groups of a certain size one can find in a collection of objects. (We'll go back to dots for now.)

For example, the computation $20 \div 5$ asks:

How many groups of five can you find among 20 *dots?*

We see four groups of five:



$$20 \div 5 = 4 .$$

We check this computation by seeing that 4×5 (four groups of five) really is 20.

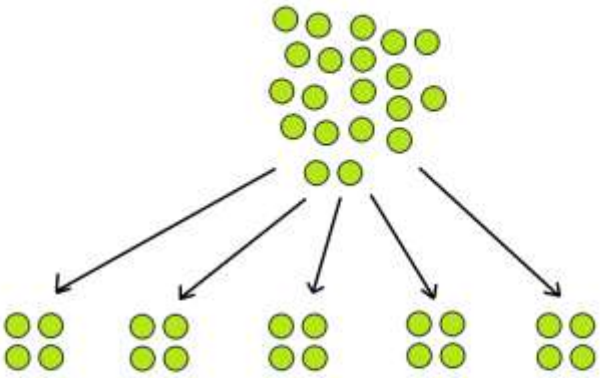
Cool Word: The division symbol $\div$ has a lovely official name. It is called an *obelus*.

DIVISION AS SHARING

Now let's ask:

I have 20 *pies. I'd like to share them (equally) among* 5 *people. How many pies per person does this give?*

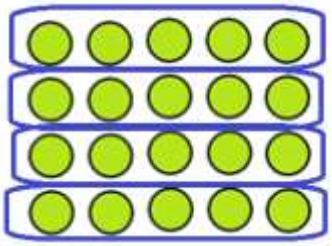
If we split the twenty piles into five equal piles, we obtain the picture:



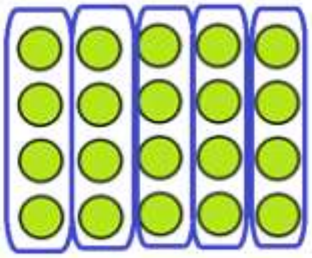
We see we get four pies per person. We write: $20 \div 5 = 4$.

These two approaches give very different pictures with the same final answer. Is it philosophically obvious that they should?

The key, again, is to first arrange dots in a rectangular array:



Finding groups of five



Sharing among five

On the left we see that the number of rows in the array gives the count of groups of five in the entire array.

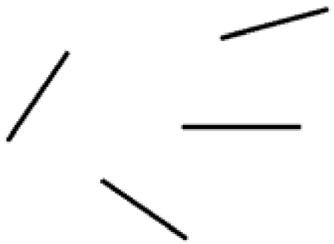
On the right we see that the number of rows in the array gives the count of how many dots (pies) are given per person.

Each interpretation counts the number of rows in the array and so each interpretation of division gives the same answer.

This is subtle!

NUMBERS ON THE NUMBER LINE

Instead of counting dots or unit squares or pies, let's imagine counting line segments each one unit long. Here's a picture of 5 unit segments:



It seems natural to arrange these along a line:

We can mark, with whole numbers, the counts of segments that line up to reach locations along the line. The line segments are all placed to the right of a special point labeled "0" on the line. (Is that point actually <u>counting</u> zero line segments?)

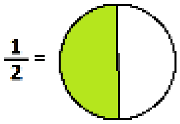
People like to think of these numbers as specifying actual points on the line. The point labeled "5," for example, is the point on the line five units to the right of the point labeled 0. The line with points labeled this way is called the <u>number line</u>.

Upshot: People like to associate with numbers points on a line. The point associated with number a is the point a units to the right of a special location labeled 0.

FRACTIONS: Their First Appearance

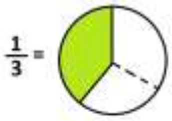
Fractions are usually first introduced to youngsters as division by sharing:

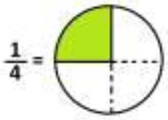
> "A half a pie is the result of dividing that pie into two equal pieces."

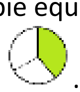


(It is presumed that the other piece of the pie has been shuffled elsewhere.)

And we draw pictures for a third of a pie, and a quarter of a pie, and so on, similarly:

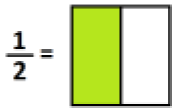


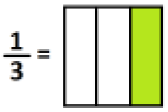


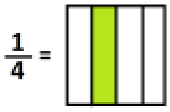
These pictures really are the results of sharing: If we share a pie equally between three people, then each person gets a piece of pie that really does look like  .

Comment: People also talk of "parts of a whole" at this point too: each action of sharing does give a part of a whole.

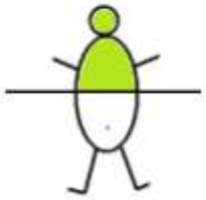
People usually draw round pies. Square or rectangular shaped pies are actually easier to draw and to think with.





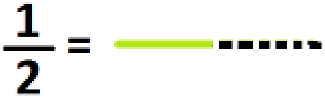


Of course one can draw pies of any shape one likes, or work with objects other than pie. For example, here is a picture of half a boy!

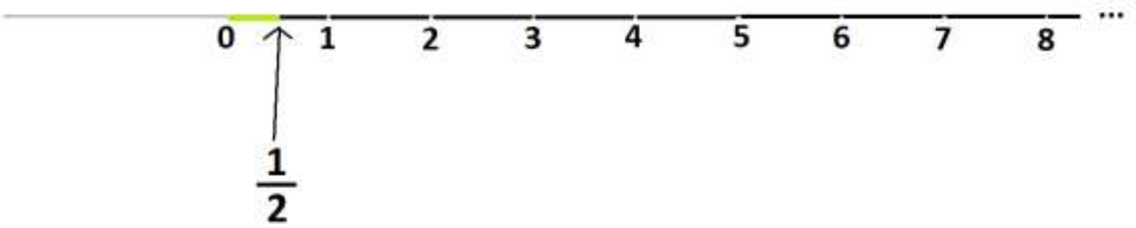


Question: Would it have been better to draw a vertical line divide the boy into two halves? (Maybe I was focusing on the height of the boy, not his volume.)

If we decide to divide unit line segments, then there is a natural place for these fractions on the number line. Here's half a line segment:



and here is the matching point on the number line:



A DIFFICULTY – OR AT LEAST SOMETHING ODD

We have "slipped in" the notation $\frac{1}{n}$ for a pie divided into n equal parts. We call each part an n th (pronounced "enth.") So…

What does something like $\frac{7}{3}$ mean?

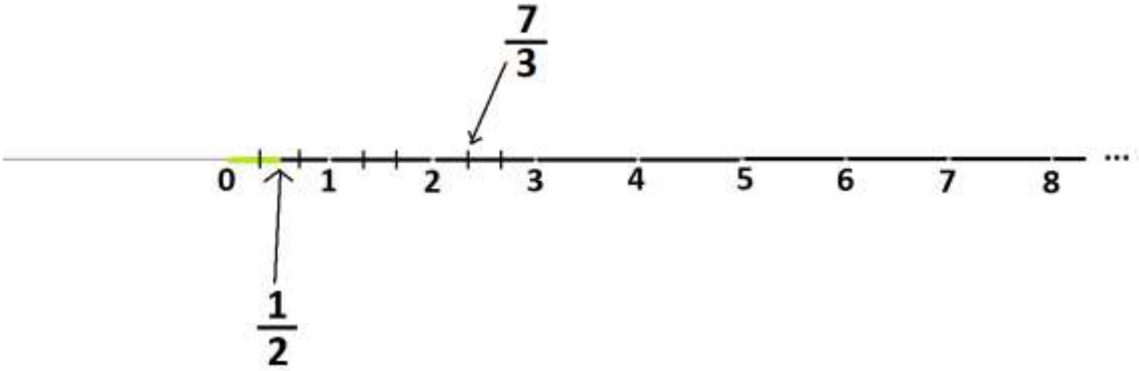
Young students are taught to interpret a quantity like this how we adults naturally read it:

$$\begin{aligned}\frac{7}{3} = \text{“seven thirds”} &= \text{seven of these: } \frac{1}{3}\\ &= \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\\ &= \text{seven groups of } \frac{1}{3}\\ &= 7\times\frac{1}{3}\ .\end{aligned}$$

In general, if a and b are each counting numbers, then $\frac{a}{b}$ is understood to mean $a\times\frac{1}{b}$ (which reads as "a copies of a beeth"?)

Note only fractions of the form $\frac{1}{n}$ are being defined as parts of a whole.

With this interpretation, these quantities $\frac{a}{b}$ have places on the number line too. For example, here's the place of $\frac{7}{3}$. It comes from stacking seven thirds together.



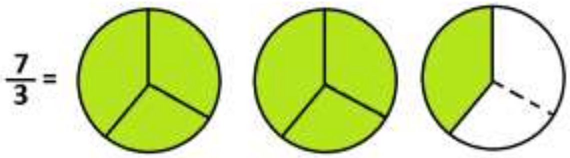
Because each quantity $\frac{a}{b}$ has a place on the number line, people like to think of these quantities as actual numbers. They call them fractions.

The potential oddness:

We started with fractions of the form $\frac{1}{n}$ that come from sharing, and then moved away from sharing to create fractions form $\frac{a}{b}$ for a larger than 1. So more general fractions aren't related to sharing?

Let's make this specific:

Back in terms of pie, here's what $\frac{7}{3}$ *(seven thirds) looks like:*

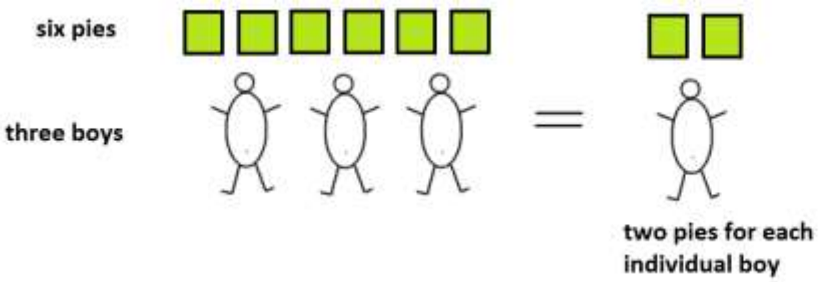


Are we saying, or at least implying, that this is unrelated to a sharing problem?

FRACTIONS: A Second Appearance

Let's push the "division as sharing" model of fractions as far as we can and see how far we can get with it. We start by developing some notation for sharing.

Example: Suppose 6 pies are to be shared equally among 3 boys. This clearly yields two pies per boy.



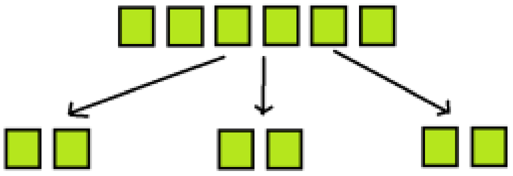
We write:

$$\frac{6}{3} = 2 .$$

We might also write $6 \div 3 = 2$ or $\begin{array}{r} 2 \\ 3\overline{)\,6} \end{array}$.

Notice that the obelus itself, $\div$, is derived from the notation $\frac{a}{b}$!

Comment: Using boys in this model is fun, concrete, and pedagogically helpful, but actually immaterial. By $\frac{6}{3}$ we actually mean "the result of sharing six pies three ways." The answer is a quantity of pie, the quantity per share, which, in this case is 2 pies.



We could draw pictures without the boys if we like, but drawing boys is fun and motivating.

We just need to keep in mind that our answers to sharing problems are always actual amounts of pie (and these amounts can be shared again, if we so wish!)

In this context …

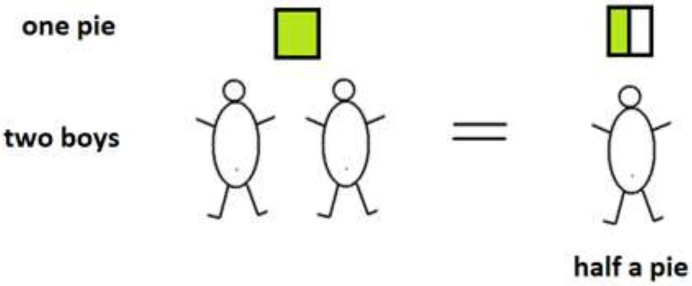
sharing 10 pies among 2 boys yields: $\frac{10}{2} = 5$ pies (per boy),

sharing 8 pies among 2 boys yields: $\frac{8}{2} = 4$ pies,

sharing 5 pies among 5 boys yields: $\frac{5}{5} = 1$ pie,

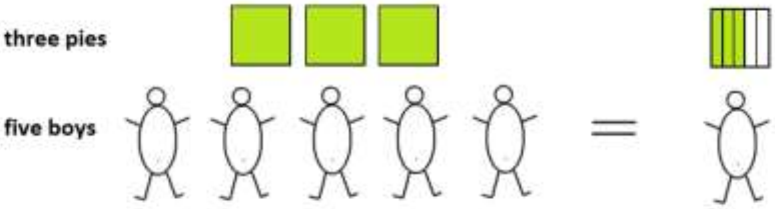
and

the answer to sharing 1 pie among 2 boys is $\frac{1}{2}$, which we call "one half.



In general, we call $\frac{1}{n}$, the result of sharing 1 pie among n boys, "one nth."

Now think about $\frac{3}{5}$. In our sharing model this is the answer to sharing three pies equally among five boys:



How might one physically accomplish task?

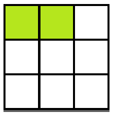
One could divide each of the five pies into five equal pieces (fifths) and give one piece from each pie to each boy. This approach gives each boy three $\frac{1}{5}$s. For this reason, we call $\frac{3}{5}$ "three fifths."

Comment: This matches the previous approach of regarding $\frac{3}{5}$ as $3 \times \frac{1}{5}$, but our thinking here is fully aligned with the one sharing model.

Some thinking questions:

Question 1:

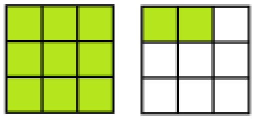
a) Here is the answer to a sharing problem:



This represents the amount of pie an individual boy receives if some number of pies is shared among some number of boys.

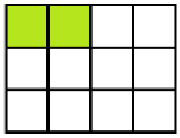
How many pies? __________
How many boys? __________
(Are answers here unique?)

b) Here is the answer to another sharing problem:



How many pies? __________
How many boys? __________
(Are answers here unique?)

c) Here is the answer to a sharing problem with non-square pie:



How many pies? __________
How many boys? __________
(Are answers here unique?)

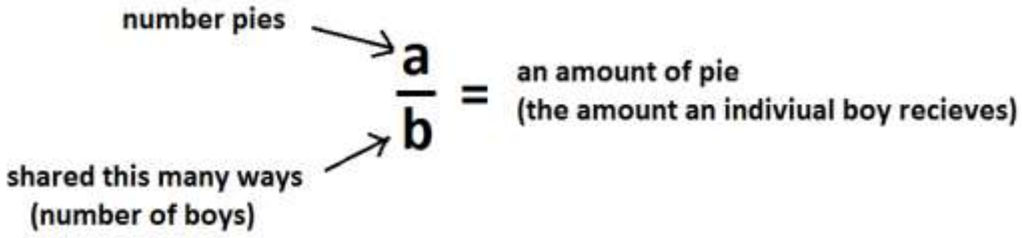
Question 2: Leigh says that "$\frac{10}{13}$ is two times as big as $\frac{5}{13}$." She argues:

In one room, ten pies are shared among thirteen boys.
In another room, five pies are shared among thirteen boys.
Each boy in the first room receives twice as much pie as each boy in the second room.

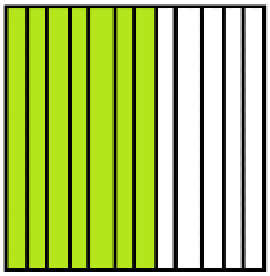
Do you agree?

Question 3: What does the sharing problem $\frac{1}{1}$ represent? How much pie does an individual boy receive? What sharing problem does $\frac{5}{1}$ represent and what is its result?

In our sharing model, $\frac{a}{b}$ represents the amount of pie an individual boy receives when a pies are shared equally among b boys.



Given a picture of an amount of pie each boy receives in a sharing problem, we can deduce the details of the sharing problem. For example, we can see seven-twelfths in this picture:



That it, this is a picture of $\frac{7}{12}$.

Jargon: In a sharing problem expressed in the form $\frac{a}{b}$ we call quantity a the <u>numerator</u> of the expression and the quantity b its <u>denominator</u>.

Question 3 leads us to say:

OBSERVATION 1: $\frac{a}{a} = 1$

OBSERVATION 2: $\frac{a}{1} = a$

(at least for any whole positive number a, but perhaps for other types of numbers too!)

Question 4: "I have no pies to share among seven boys." Use this to make a mathematical statement about a sharing problem.

A SUBTLE POINT:

Notice that I am avoiding the use of the word "fraction" for the moment. If a and b are positive whole numbers, people will call $\frac{a}{b}$ a <u>fraction</u>. But there might be (and, in fact, are) instances when a and b are not positive whole numbers and we might still want to call $\frac{a}{b}$ a fraction. Let's just be slow and methodical with our thinking for now and consider all possibilities that might arise and be of interest before locking ourselves into a strict definition.

Thinking Question 5: Going backwards …

Suppose I told you in a sharing problem with seven boys, each boy received three pies. How many pies, in total, were there to begin with?

$$\frac{??}{7} = 3$$

Logic tells us that there were 21 pies. Is it a coincidence that $3 \times 7 = 21$?

In general, if $\frac{a}{b} = n$, must it be the case that $a = n \times b$?

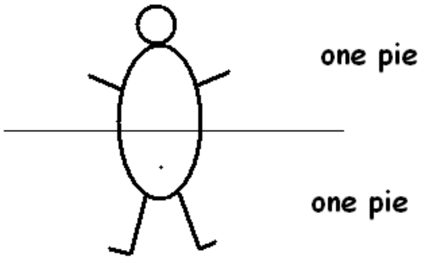
HAVING FUN WITH THE NUMERATORS AND DENOMINATORS

Let's be wild!

Let's make sense of the sharing problem $\frac{1}{\left(\frac{1}{2}\right)}$*, sharing one pie among half a boy!*

Now $\frac{a}{b}$, in general, represents the amount of pie each boy receives in a sharing problem. That is, the amount of pie a full individual boy receives.

In $\frac{1}{1/2}$ we are providing one pie for half a boy. So if each half is assigned one pie, how much pie per (whole) boy is that? Answer: Two pies!



We have:

$$\frac{1}{\left(\frac{1}{2}\right)} = 2 .$$

Whoa!

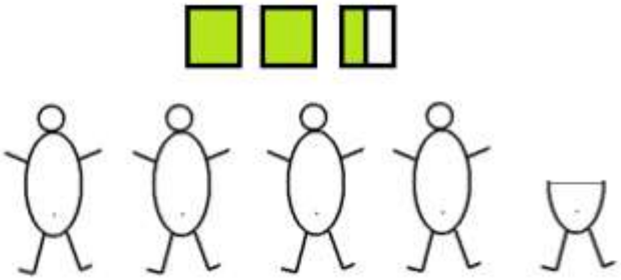
In the same way, distributing one pie to each third of a boy yields 3 pies for an individual boy:

$$\frac{1}{\left(\frac{1}{3}\right)} = 3 .$$

And distributing five pies for every seventh of a boy yields a total of 35 pies for a full boy:

$$\frac{5}{1/7} = 35 .$$

Challenge question 6:
Two-and-a-half pies are to be shared equally among four-and-a-half boys! How much pie does an individual (whole) boy receive?



It is possible to think our way through this right now, but it is tricky. (Can you develop, right now, a philosophically swift way to see through this?)

COMPARISON MOMENT

Recall from page 6 that there are two different ways to interpret division. We've just interpreted $\frac{5}{1/7}$ as "division as sharing," that is, *share five pies over one-seventh of a boy*. In the number-line model of fractions, one makes sense of $\frac{5}{1/7}$ as "division as groups": *how many one-seventh segments can I find in the segment of length five?* We see there are 35.

Can you see how the number-line model give $\frac{1}{1/n} = n$?

THE KEY FRACTION PROPERTY:

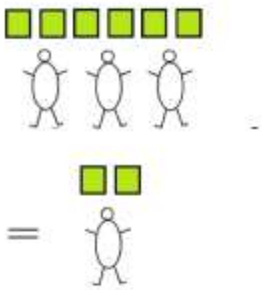
We have that $\frac{a}{b}$ is an answer to a division problem:

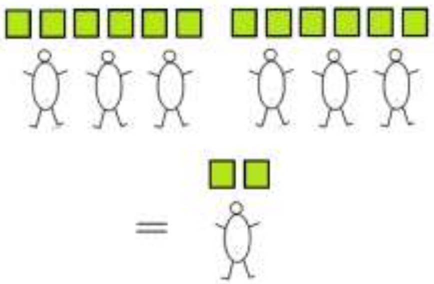
> $\frac{a}{b}$ *represents the amount of pie an individual boy receives when* a *pies are distributed among* b *boys.*

What happens if we double the number of pies and double the number of boys? Nothing! The amount of pie per boy is still the same:

$$\frac{2a}{2b} = \frac{a}{b}.$$

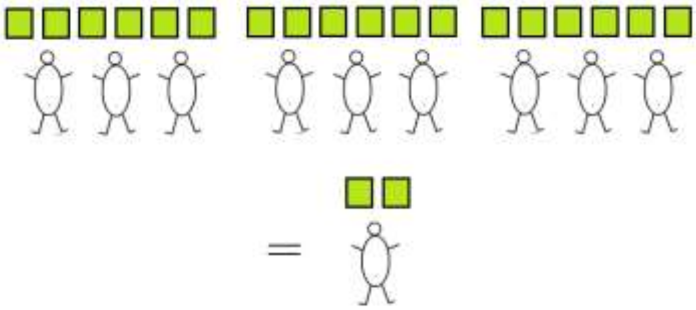
For example, this picture shows that $\frac{6}{3}$ and $\frac{12}{6}$ both give two pies to each boy:





And tripling the number of pies and tripling the number of boys also does not change the final amount of pie per boy, nor does quadrupling each number, or one-trillion-billion-tupling the numbers!

$$\frac{6}{3} = \frac{12}{6} = \frac{18}{9} = \cdots \text{ = two pies per boy}$$



This leads us to want to believe a rule:

KEY FRACTION PROPERTY 3: $\frac{xa}{xb}=\frac{a}{b}$
(at least for positive whole numbers – but maybe for other types of numbers too!).

For example,

$$\frac{3}{5}$$ (sharing three pies among five boys)

yields the same result as

$$\frac{3\times 2}{5\times 2}=\frac{6}{10}$$ (sharing six pies among ten boys),

and as

$$\frac{3\times 100}{5\times 100}=\frac{300}{500}$$ (sharing 300 pies among 500 boys).

Going backwards …

$$\frac{20}{32}$$ (sharing 20 pies among 32 boys)

is the same problem as:

$$\frac{5\times 4}{8\times 4}=\frac{5}{8}$$ (sharing five pies among eight boys).

Comment: Most people say we have “cancelled” a common factor of 4 from the numerator and the denominator and, in doing so, we have “simplified” the expression $\frac{20}{32}$. (Admittedly, $\frac{5}{8}$ is easier to conceptualize than $\frac{20}{32}$.)

As another example $\frac{280}{350}$ can certainly be made to look more manageable by noticing that there is a common factor of 10 in both the numerator and the denominator:

$$\frac{280}{350} = \frac{28 \times 10}{35 \times 10} = \frac{28}{35}.$$

We can go further as 28 and 35 are both multiples of 7:

$$\frac{28}{35} = \frac{4 \times 7}{5 \times 7} = \frac{4}{5}.$$

Thus, sharing 280 pies among 350 boys gives the same result as sharing just 4 pies among 5 boys: much easier to conceptualize!

$$\frac{280}{350} = \frac{4}{5}.$$

As 4 and 5 share no common factors, this is as far as we can go with this example (while staying with whole numbers!).

Question 7: Jenny says that $\frac{4}{5}$ does "simplify" further is you are willing to move away from whole numbers. She writes:

$$\frac{4}{5} = \frac{2 \times 2}{2\frac{1}{2} \times 2} = \frac{2}{2\frac{1}{2}}.$$

Is she right? Does sharing 4 pies among 5 boys yield the same result as sharing 2 pies among $2\frac{1}{2}$ boys? What do you think? (And is her expression "simpler" than the original?)

Comment: We'll talk about mixed numbers, such as $2\frac{1}{2} = 2 + \frac{1}{2}$, formally later on.

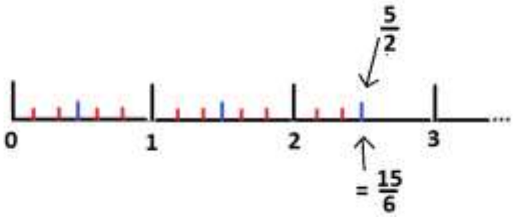
COMPARISON MOMENT

How is the key fraction property explained in the number-line module?

The following is understood from practice examples: Consider a picture of a copies of segments length $\frac{1}{b}$ stacked together landing at position $\frac{a}{b}$ on the number line. If we divide each of those small segments into x equal parts, then we have a picture of xa copies of segments of length $\frac{1}{xb}$ stacked together landing at the same position. That is, we have:

$$\frac{xa}{xb} = \frac{a}{b} .$$

For example, here is a picture of five copies of $\frac{1}{2}$, each copy divided into $x = 3$ parts:



One issue: There is a technical point one should think through at least once:

> *Does dividing a line segment represented by length* $\frac{1}{b}$ *into* x *equal parts give a line segment that deserves to be called* $\frac{1}{xb}$*?*

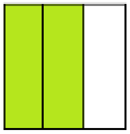
The number-line model defined $\frac{1}{b}$ to be the line segment that results when dividing a unit segment into b equal parts. So b copies of $\frac{1}{b}$ should stack together to make a unit line segment. (This is how one checks that a candidate length for $\frac{1}{b}$ is correct.)

Now divide a segment of length $\frac{1}{b}$ into x equal parts. Call one of those small lengths L. We have that x copies of L stack to the length $\frac{1}{b}$, and b copies of $\frac{1}{b}$ stack to make a copy of a unit length. It follows then that xb copies of L stack to make a unit length and so L does indeed deserve to be called $\frac{1}{xb}$.

ANOTHER APPROACH TO THE KEY FRACTION PROPERTY:

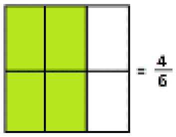
Motivated by the early-grade number-line model we can lead in to the Key Fraction Property a second way. We'll start with a concrete example.

Here is a picture of some pie after sharing exercise:

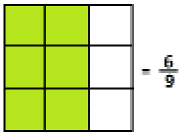


We see it is a picture of $\frac{2}{3}$, sharing two pies equally among three girls. (It's time that girls got some pie too!)

But draw an extra cut in the picture, and we now see four-sixths, the result of sharing four pies among six girls.



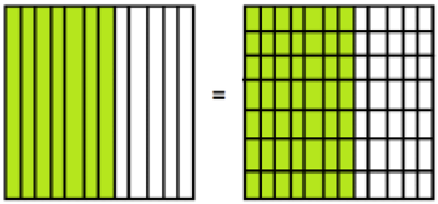
We haven't changed the amount of pie, so $\frac{4}{6}=\frac{2}{3}$. Or if we divide the amount of pie into three equal parts we see: $\frac{6}{9}=\frac{2}{3}$:



In general, dividing the picture of $\frac{2}{3}$ into x equal parts we'll see $\frac{2x}{3x}=\frac{2}{3}$.

And of course there is nothing special about the amount $\frac{2}{3}$ to begin with.

In general, we have: $\frac{xa}{xb}=\frac{a}{b}$.



SO … WHAT IS A FRACTION?

We are ready now to set what we mean by a fraction:

We have the Key Property of Fractions: $\frac{xa}{xb} = \frac{a}{b}$, at least for positive whole numbers. But let's allow x to be <u>any</u> type of number.

> **Definition:** A <u>fraction</u> is any answer to a division problem that is equivalent, via the Key Fraction Property, to a division problem $\frac{a}{b}$ with a and b whole numbers.

For example, $\frac{2}{3}$ is a fraction (it is already of good form) and so is $\frac{1.2}{3}$ (by the Key Fraction Property this is equivalent to $\frac{1.2\times 10}{3\times 10} = \frac{12}{30}$), as is $\frac{57\sqrt{2}}{61\sqrt{2}}$ (which is equivalent to $\frac{57}{61}$).

In our model, fractions are amounts of pie. And so are whole numbers!
For example, "2" is two pies: 🟩🟩.

Since we like to believe counts of objects, in our case pies, are numbers in their own right:

> **Fractions, like the counting numbers, are numbers too.**

We can locate the position of fractions on the number line, if we wish, just as the number-line model from grade school does.

But really, as answers to division problems, fractions are the numbers that are answers to arithmetic division problems.

ADDING AND SUBTRACTING FRACTIONS:

If we think of fractions as amounts of pie that arise from sharing problems, then it makes sense, in that model, to add and subtract amounts of pie and get an answer that it the result of another sharing problem.

For example, here is a picture of $\frac{2}{3}+\frac{2}{3}$:

The result is four-thirds, the result of sharing four pies equally among three boys.

$$\frac{2}{3}+\frac{2}{3}=\frac{4}{3}.$$

In the same way we could draw a picture of $\frac{2}{7}+\frac{3}{7}$ and see the answer $\frac{5}{7}$.

This really is as easy as saying “two sevenths plus three sevenths makes five sevenths,” just as two apples plus three apples makes five apples.

And we can add more than two fractions sharing a common denominator with the same ease:

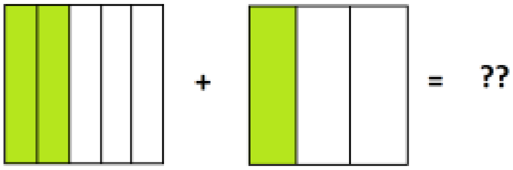
$$\frac{1}{10}+\frac{8}{10}+\frac{4}{10}=\frac{13}{10}$$

and subtract fractions with a common denominator:

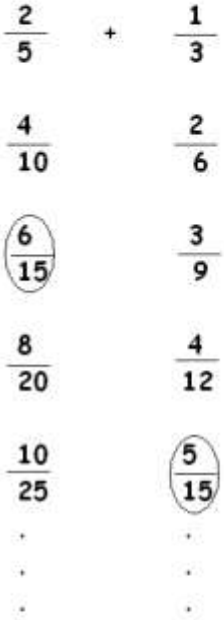
$$\frac{6}{11}-\frac{2}{11}=\frac{4}{11}.$$

The rub comes in trying to add or subtract fractions with different denominators.

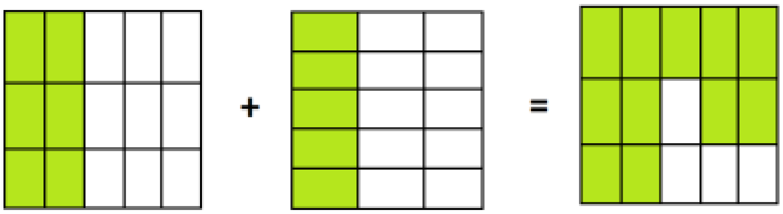
Example: As a sharing problem, what is the result of $\frac{2}{5}+\frac{1}{3}$?



But we can rewrite each of the expressions $\frac{2}{5}$ and $\frac{1}{3}$ in an infinitude of equivalent forms using the Key Fraction Property. By luck, we might find two expressions with a common denominator:



We do luck out and see that $\frac{2}{5}+\frac{1}{3}$ is actually equivalent to the problem $\frac{6}{15}+\frac{5}{15}$, which has answer $\frac{11}{15}$. This work shows we should slice each of the two amounts of pie we have in our picture into fifteenths and all should then be clear. (And it is!)



$$\frac{2}{5}+\frac{1}{3} \quad = \quad \frac{6}{15}+\frac{5}{15} \quad = \quad \frac{11}{15}$$

As another example:

$$\begin{array}{ccccc}
\frac{3}{8} & + & \frac{3}{10} & = & \frac{27}{40} \\[2ex]
\frac{6}{16} & & \frac{6}{20} & & \\[2ex]
\frac{9}{24} & & \frac{9}{30} & & \\[2ex]
\frac{12}{32} & & \boxed{\frac{12}{40}} & & \\[2ex]
\boxed{\frac{15}{40}} & & \frac{15}{50} & & \\[2ex]
\vdots & & \vdots & &
\end{array}$$

Of course, there is an efficient way to find a "common denominator" of two fractions. You've no doubt thought of it already.

Question 8: a) What is $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$?

b) What is $\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}$?

Here's a good question!

Question 9: Which is larger: $\frac{5}{9}$ or $\frac{6}{11}$?

A FIRST STEP TO MULTIPLICATION:

In a sharing problem, how might you double the amount of pie each girl receives?

Answer: Double the number of pies!

Let's translate this our mathematical notation:

$\frac{a}{b}$ is the amount of pie each girl receives in the sharing problem

"share a pies among b girls."

$2\times\frac{a}{b}$ is "two groups" of this amount of pie, that, $2\times\frac{a}{b}$ is double this amount of pie.

But we just concluded that we can double the amount of pie per girl by sharing $2a$ pies among the b girls instead. Thus we conclude:

$$2\times\frac{a}{b}=\frac{2a}{b}.$$

In the same way, to triple the amount of pie each girl receives in a sharing problem, just triple the amount of pie:

$$3\times\frac{a}{b}=\frac{3a}{b}.$$

And so on.

This leads to the belief:

FIRST MULTIPLICATION BELIEF 4: $x\times\frac{a}{b}=\frac{xa}{b}$

(at least for positive whole numbers x, a, and b, but maybe for other types of numbers too!).

Comment: Motivated by our belief that multiplication should be commutative, we'll choose to interpret "$\frac{a}{b}\times x$" as $x\times\frac{a}{b}$.

COMPARISON MOMENT

In the number-line model, $\frac{a}{b}$ is defined to mean $a \times \frac{1}{b}$: the endpoint of a copies of segments $\frac{1}{b}$ stacked together. So $x \times \frac{a}{b}$ is technically $x \times \left(a \times \frac{1}{b} \right)$. Is this the same as $\frac{xa}{b}$? (Is it obvious, for example, that five copies of two copies of $\frac{1}{3}$, that is, $5 \times \frac{2}{3}$, is ten copies of $\frac{1}{3}$, that is, $\frac{10}{3}$?)

There is a nice consequence of our first multiplication belief:

CONSEQUENCE 5: $b \times \frac{a}{b} = a$

(at least for positive whole numbers a, and b, but maybe for other types of numbers too).

Reason:

$$\begin{aligned} b \times \frac{a}{b} &= \frac{ba}{b} && \text{(by our first multiplication belief)} \\ &= \frac{a \times b}{1 \times b} && \text{(by standard arithmetic)} \\ &= \frac{a}{1} && \text{(by the key fraction property)} \\ &= a && \text{(by observation 2).} \end{aligned}$$

For example:

$$3 \times \frac{7}{3} = 7 \qquad \text{and} \qquad \frac{853}{76} \times 76 = 76 \times \frac{853}{76} = 853.$$

THE WORD "OF":

In the world of fractions the word "of" is often used to imply sharing. But it is confusing.

For example, by "*one-third of six*" we mean "share six objects among three boys." The result is two of those objects. That is, one-third of six is interpreted as $\frac{6}{3}$.

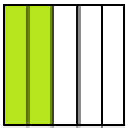
In general:

> *By "$\frac{1}{n}$ of a" we mean the sharing problem $\frac{a}{n}$, the result of sharing the quantity a among n people.*

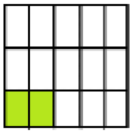
This is the case even if a itself is a fraction!

Example: *What is $\frac{1}{3}$ of $\frac{2}{5}$?*

Answer: Here we are being asked to share the quantity $\frac{2}{5}$ among three girls. In terms of pie, here is a picture of $\frac{2}{5}$:



Here is the result, for each girl, of sharing it equally among the three of them:



We see that this is a picture of $\frac{2}{15}$.

Of course, we just said that "$\frac{1}{n}$ of a" is to be interpreted as $\frac{a}{n}$. We see we get the same answer if we just blindly follow the arithmetic:

$$\frac{1}{3} \text{ of } \frac{2}{5} = \frac{2/5}{3} = \frac{(2/5)\times 5}{3\times 5} = \frac{2}{15}.$$

Popular language has us interpret “two-thirds of six” as “two copies of one-third of six.” This is thus four objects out of a collection of six objects.

In general:

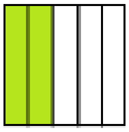
By “$\frac{x}{n}$ of a” we mean “x copies of $\frac{a}{n}$,” that is, $x \times \frac{a}{n}$.

And, again, this is the case even if a itself is a fraction.

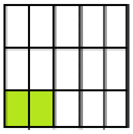
Example: *What is $\frac{2}{3}$ of $\frac{2}{5}$?*

Answer: This time we want two copies of one-third of two-fifths!

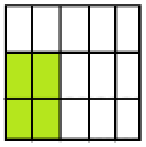
Here’s $\frac{2}{5}$:



Here’s one-third of $\frac{2}{5}$.



Here’s two copies of one-third of $\frac{2}{5}$. We see it is $\frac{4}{15}$.

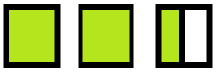


Of course, we just said that “$\frac{x}{n}$ of a” is to be interpreted as $x \times \frac{a}{n}$. So blind arithmetic gives:

$$\frac{2}{3} \text{ of } \frac{2}{5} = 2 \times \frac{2/5}{3} = 2 \times \frac{2}{15} = \frac{4}{15}.$$

MIXED NUMBERS:

By “two and a half pies” we mean two pies and another half a pie: $2+\frac{1}{2}$.



Even though we say the word “and” out loud, we tend to omit the plus sign in our mathematical writing. Thus $2+\frac{1}{2}$ is usually written $2\frac{1}{2}$.

Sharing more pies than there are girls gives answers that can be expressed as mixed numbers.

For example, if seven pies are shared among three girls, each girl will receive two whole pies and a third of a pie. We have:

$$\frac{7}{3}=2\frac{1}{3}.$$

As another example, consider $\frac{23}{4}$. We can certainly give each girl five whole pies. That leaves three pies still be shared among the four girls. We see:

$$\frac{23}{4}=5\frac{3}{4}.$$

Comment: Most people say: “Four goes into 23 five times and leaves a remainder of 3, which is still be divided by four.”

CHECKING OUR RESULTS:

By consequence 5 we have $\frac{7}{3}\times 3=7$. As a check, does $2\frac{1}{3}$ fit this equation too?

$$2\frac{1}{3}\times 3=\left(2+\frac{1}{3}\right)\times 3=6+\frac{1}{3}\times 3=6+1=7.$$

Yes!

Is $5\frac{3}{4}$ indeed $\frac{23}{4}$?

$$5\frac{3}{4}\times 4=\left(5+\frac{3}{4}\right)\times 4=20+3=23\ .$$

Yes!

Question 10: JinPyo says that the mixed number $5\frac{6}{5}$ is actually $\frac{31}{5}$. Do you agree?

Question 11: Write each of the following as a mixed number.

a) $\frac{8}{5}$ b) $\frac{100}{13}$ c) $\frac{200}{199}$ d) $\frac{199\frac{1}{2}}{199}$

(Is question d) meaningful? Should we have a strict definition of what a "mixed number" should be?)

We can go backwards…

Example: *What sharing problem with a whole number of pies and a whole number of girls gives the answer* $3\frac{5}{7}$*?*

Thinking-through-it Answer: There is a clue in this question: we can see that we are sharing among seven girls.

Each of those girls gets three whole pies and five-sevenths of a pie. The five-sevenths must come from five whole pies being shared among the girls. There must have been a total of $3\times 7+5=26$ pies.

We have: $3\frac{5}{7}=\frac{26}{7}$.

The-mechanics-of-fractions Answer:

We really have the addition problem $3+\frac{5}{7}$ at hand here. This is best analyzed by working with fractions with a common denominator:

$$3+\frac{5}{7}=\frac{3}{1}+\frac{5}{7}=\frac{21}{7}+\frac{5}{7}=\frac{26}{7}.$$

Example: *What sharing problem with a whole number of pies and a whole number of pies has the answer* $1\frac{2}{11}$*?*

Answer: $1+\frac{2}{11}=\frac{11}{11}+\frac{2}{11}=\frac{13}{11}$.

Question 12: Give three different sharing problems, each with a whole number of pies and a whole number of boys, which each give the answer $200\frac{1}{200}$.

Old Jargon:

A fraction with numerator smaller than denominator used to be called a proper fraction. For example, $\frac{45}{85}$ is a proper fraction.

A fraction with numerator larger than denominator used to be called an improper fraction. For example, $\frac{7}{3}$ is a proper fraction. (In the 1800s these fractions were call vulgar fractions. They were considered "common.")

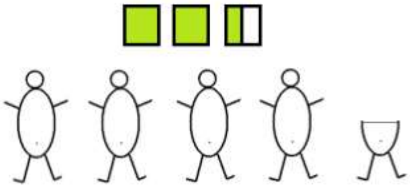
In curricula materials of recent decades, students were always expected to write improper fractions as mixed numbers. The reasons for this are not clear. (Though one might argue that $2\frac{1}{3}$ feels more manageable conceptually than $\frac{7}{3}$.)

DIVISION OF FRACTIONS:

As the quantities $\frac{a}{b}$ are defined through the notion of “division as sharing” it is not at all surprising that the operation of division is straightforward and natural for these quantities, especially with the key fraction property in hand.

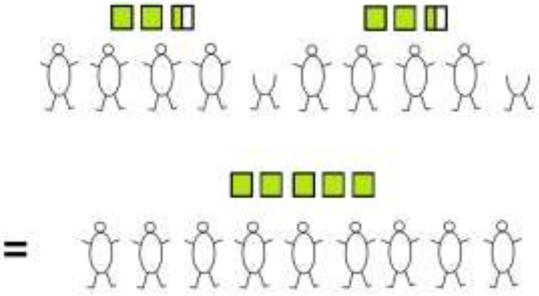
Let’s return to the challenge example of page 18.

> *Two-and-a-half pies are to be shared equally among four-and-a-half boys! How much pie does an individual (whole) boy receive?*



We’re being asked to make sense of: $\frac{2\frac{1}{2}}{4\frac{1}{2}}$.

Epiphany: If we double the number of pies and double the number of boys, then we see that this sharing task is equivalent to the sharing task $\frac{9}{5}$. That is, each boy ends up with nine-fifths of a pie.



We have: $\frac{4\frac{1}{2}}{2\frac{1}{2}} = \frac{\left(4+\frac{1}{2}\right)\times 2}{\left(2+\frac{1}{2}\right)\times 2} = \frac{9}{5}$.

Comment: We need to be flexible about how two halves of boys combine to make the equivalent of a whole boy!

So we see that sharing $4\frac{1}{2}$ equally among $2\frac{1}{2}$ is equivalent to sharing 9 pies among 5 boys. Each boy thus receives $1\frac{4}{5}$ pies.

Another example:

> $7\frac{2}{3}$ *pies are to be shared among* $5\frac{3}{4}$ *girls. How many pies does each individual girl receive?*

We want to make

$$\frac{7+\frac{2}{3}}{5+\frac{3}{4}}$$

more manageable. Using the Key Fraction Property, let's multiply the numerator and denominator of this fraction each through by 3. (Why three?) We'll make use of our multiplication rule: $x\times\frac{a}{b}=\frac{xa}{b}$ along the way:

$$\frac{\left(7+\frac{2}{3}\right)\times 3}{\left(5+\frac{3}{4}\right)\times 3}=\frac{21+2}{15+\frac{9}{4}}.$$

Let's now multiply numerator and denominator each by 4. (Why four?)

$$\frac{(21+2)\times 4}{\left(15+\frac{9}{4}\right)\times 4}=\frac{84+8}{60+9}=\frac{92}{69}.$$

This shows that sharing $7\frac{2}{3}$ pies among $5\frac{3}{4}$ girls is equivalent to sharing 92 pies among 69 girls. (Is that friendlier?) Each girl receives $1\frac{23}{69}$ pies.

As another example, consider $\dfrac{3\frac{1}{2}}{1\frac{1}{2}}$.

Multiplying the numerator and denominator each by 2 should be enough to make the expression look friendlier:

$$\frac{3\frac{1}{2}}{1\frac{1}{2}}=\frac{3+\frac{1}{2}}{1+\frac{1}{2}}=\frac{\left(3+\frac{1}{2}\right)\cdot 2}{\left(1+\frac{1}{2}\right)\cdot 2}=\frac{6+1}{2+1}=\frac{7}{3}.$$

Now look at each of these examples. What would you do to make each expression look more manageable?

a) Make $\dfrac{4\frac{2}{3}}{5\frac{1}{3}}$ look friendlier.

b) Make $\dfrac{2\frac{1}{5}}{2\frac{1}{4}}$ look friendlier.

c) Make $\dfrac{1\frac{4}{7}}{2\frac{3}{10}}$ look friendlier.

d) Make $\dfrac{\frac{3}{5}}{\frac{4}{7}}$ look friendlier.

Let's work through the fourth one here.

Multiply numerator and denominator each by 5:

$$\frac{\frac{3}{5}\times 5}{\frac{4}{7}\times 5}=\frac{3}{\frac{20}{7}}$$

Now multiply the numerator and denominator each by 7:

$$\frac{3\times 7}{\frac{20}{7}\times 7}=\frac{21}{20}.$$

That looks more manageable.

Comment: Most people would say that we have just computed $\frac{3}{5} \div \frac{4}{7}$ as $\frac{21}{20}$.

As another example, let's compute $\frac{5}{9} \div \frac{8}{11}$, that is, let's make:

$$\frac{\frac{5}{9}}{\frac{8}{11}}$$

more manageable.

> Multiply numerator and denominator each by 9 and by 11 at the same time. (Why not?)
>
> $$\frac{\frac{5}{9} \times 9 \times 11}{\frac{8}{11} \times 9 \times 11} = \frac{5 \times 11}{8 \times 9}$$

(Do you see what happened here?)

and so:

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{5 \times 11}{8 \times 9} = \frac{55}{72}.$$

Question 13: Compute each of the following:

a) $\frac{1}{2} \div \frac{1}{3}$ b) $\frac{4}{5} \div \frac{3}{7}$ c) $\frac{2}{3} \div \frac{1}{5}$

d) Abstractly compute $\frac{a}{b} \div \frac{c}{d}$.

Notice that dividing mixed numbers first naturally slides us into the division of fractions with no fuss.

Question 14: Compute $\frac{45}{45} \div \frac{902}{902}$. Do you see what the answer simply must be?

Question 15: Compute $\frac{10}{13} \div \frac{2}{13}$. Any general comments about this one?

Question 16: Some teachers have students solve division problems by having students rewrite terms to have a common denominator. For example, to compute:

$$\frac{3}{4} \div \frac{2}{3}$$

rewrite the problem as:

$$\frac{9}{12} \div \frac{8}{12}.$$

The claim is then made that the answer to the original problem is $9 \div 8 = \frac{9}{8}$.

a) Does $\frac{3}{4} \div \frac{2}{3}$ indeed equal $\frac{9}{8}$?

b) Work out $\frac{5}{4} \div \frac{7}{9}$ via the method of this section, and then again by the method described above. Are the answers indeed the same?

Why does this "common denominator method" work?

Question 17: Work out $\frac{12}{15} \div \frac{3}{5}$ and show that it equals $\frac{4}{3}$.

Now notice that

$$\begin{gathered} 12 \div 3 = 4 \\ 15 \div 5 = 3 \end{gathered}$$

and

$$\frac{12}{15} \div \frac{3}{5} = \frac{4}{3}.$$

Is this a coincidence or does $\frac{a}{b} \div \frac{c}{d}$ always equal $\frac{a \div c}{b \div d}$?

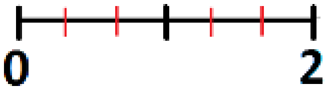
★COMPARISON MOMENT ★

In the number-line model, division of quantities $\frac{a}{b}$ and $\frac{c}{d}$ returns to the "division by finding groups" interpretation of division. Computing division is hard!

For example:

Evaluate $\frac{2}{1/3}$

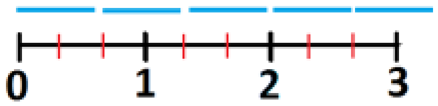
asks "How many thirds do you see in two?" The answer is six.



As examples increase in complexity, so does the thinking and visual acuity needed.

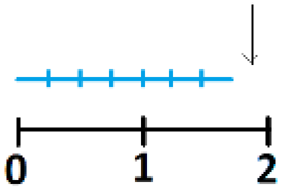
Example: *Find* $\frac{3}{2/3}$.

Answer: There are $4\frac{1}{2}$ two-thirds in a length of three: $\frac{3}{2/3} = 4\frac{1}{2}$.



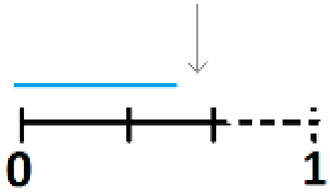
Example: *Find* $\frac{2}{1\frac{3}{4}}$.

Answer: In the picture we see that one-seventh of $1\frac{3}{4}$ is needed to fill up a gap. Thus we see one and one-seventh copies of $1\frac{3}{4}$ in 2 .



Example: *Find* $\frac{2/3}{1/2}$.

Answer: Do you see that one-sixth of the blue line of length $\frac{1}{2}$ fills the gap?



Thus we see one-and-one-sixth copies of $\frac{1}{2}$ in $\frac{2}{3}$. We have $\frac{2/3}{1/2} = 1\frac{1}{6}$.

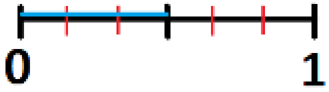
Example: *Find* $\frac{4/9}{7/5}$.

Answer: Too hard!

Curriculum writers following this number-line model for fractions are sometimes inconsistent in which interpretation of division to use in problems. (This, technically, is not a concern mathematically as both interpretations of division are valid and equivalent, but it can be confusing for students.)

Example: *Find* $\frac{1/2}{3}$.

Answer: Dividing a segment of length $\frac{1}{2}$ into three equal parts gives pieces that deserve to be called sixths.



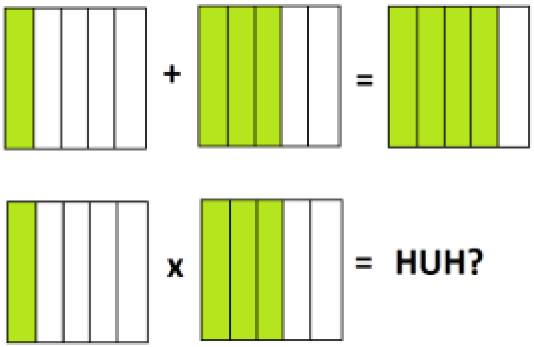
(This was "division as sharing." How does one answer this question in terms of finding groups? Can you see the answer one-sixth?)

MULTIPLICATION OF FRACTIONS:

So far we have made sense of adding and subtracting fractions, and dividing fractions. Pinning down a sensible picture of multiplying fractions is surprisingly difficult.

And this, I think, illustrates why fractions are so hard for us humans to understand and handle. Every model for fractions we use to help them internalize and understand them eventually breaks down, and it usually breaks down when it comes to multiplying fractions.

On one level we can see that it makes perfect sense to add (combine) amount of pie represented by fractions. But what does it mean to multiply pie?



Now, in our sharing model, we do have a notion of division of fractions. For example:

$$\frac{\frac{2}{5}}{\frac{3}{4}} = \frac{\frac{2}{5}\times 5\times 4}{\frac{3}{4}\times 5\times 4} = \frac{8}{15} .$$

Since division is, arithmetically, the reverse of multiplication, this calculation shows that the product of $\frac{8}{15}$ and $\frac{3}{4}$, whatever that means, should be $\frac{2}{5}$.

In fact, we can use this general idea to develop an exceedingly unfriendly and unintuitive formula what the product $\frac{a}{b}\times\frac{c}{d}$ should be. If you game to read it, here's the unfriendly argument:

Suppose $\frac{a}{b}\times\frac{c}{d}=\frac{n}{m}$.

Then we must have $\dfrac{\frac{n}{m}}{\frac{c}{d}} = \dfrac{a}{b}$.

But the left side can be rewritten $\dfrac{nd}{mc}$, so $\dfrac{nd}{mc} = \dfrac{a}{b}$.
This means that $nd = xa$ and $mc = xb$ for some number x.
Consequently $n = \dfrac{xa}{d}$ and $m = \dfrac{xb}{c}$ and the answer $\dfrac{n}{m}$ to our product equals:

$$\frac{\frac{xa}{d}}{\frac{xb}{c}} = \frac{\frac{xa}{d} \times d \times c}{\frac{xb}{c} \times d \times c} = \frac{xac}{xbd} = \frac{ac}{bd} \quad .$$

That is, we have:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} .$$

Apparently, to multiply two fractions, just create a new fraction by multiplying numerators and multiplying denominators! (Weird!)

Comment: We can adopt an easier, purely intellectual, approach if we go with the First Multiplication Belief 4, that $x \times \dfrac{a}{b} = \dfrac{xa}{b}$, but assuming it holds for all types of numbers x, including x itself being a fraction. In this case:

$$\frac{c}{d} \times \frac{a}{b} \qquad = \frac{\frac{c}{d} \times a}{b} \qquad \text{(by Belief 4)}$$

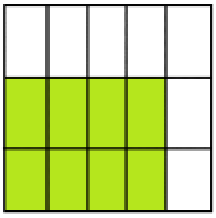
$$= \frac{\frac{ac}{d}}{b} \qquad \text{(by Belief 4)}$$

$$= \frac{\frac{ac}{d} \times d}{b \times d} \qquad = \frac{ac}{bd} .$$

This too is “mechanical” and offers no intuition as to what the formula for the product of fractions means.

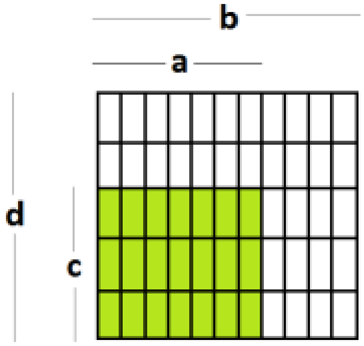
A LOVELY COINCIDENCE?

Here's a picture of "$\frac{2}{3}$ of $\frac{4}{5}$:"



We see picture of $\frac{8}{15}$. And notice that $\frac{8}{15}=\frac{2\times 4}{3\times 5}$, the product of the original numerators over the product of the original denominators.

In general, a picture of "$\frac{a}{b}$ of $\frac{c}{d}$" (that is, a beeths of c deeths) gives a picture of pie divided into a total of bd pieces with ac of them shaded.



That is, "$\frac{a}{b}$ of $\frac{c}{d}$" equals $\frac{ac}{bd}$, which is precisely what the mathematics wants the product $\frac{a}{b}\times\frac{c}{d}$ to be.

So some people just tell students:

> *Let's define the product of fractions as follows.*
> *Read*
>
> $\frac{a}{b}\times\frac{c}{d}$ as "$\frac{a}{b}$ of $\frac{c}{d}$," *a fraction of a fraction of pie.*

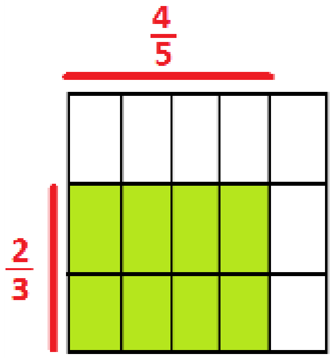
(And this propagates the idea "of" in mathematics always translates to "multiply.") This is pedagogically suspect, but, by happy coincidence, the mathematics is correct and consistent.

⭐COMPARISON MOMENT ⭐

Those working in the number-line model know that multiplying lengths, in geometry, makes perfectly good sense: it corresponds to the area of the rectangle with sides given by those two lengths.

Thus $\frac{2}{3} \times \frac{4}{5}$ is defined to be the area of the rectangle, with side lengths $\frac{2}{3}$ and $\frac{4}{5}$.

To draw this rectangle, draw the unit square, divide one side into three equal lengths and mark off $\frac{2}{3}$, divide the second side into five equal lengths and mark off $\frac{4}{5}$. Now shade the rectangle of interest:



Now comes the question: *What is the area of the shaded region?*

At this point, those working in the number-line model, having abandoned the sharing model, now return to it (!) and argue that the shaded region is given as eight-fifteenths of the unit square, and so represents $\frac{8}{15}$. (Remember: In the number-line model, fractions are lengths, but here $\frac{8}{15}$, is a proportion of area and is still being called a fraction and so is still, somehow, a length on the number line too.)

The number-line model breaks down when it comes to multiplication of fractions: it can't survive as a fully stand-alone model.

Where we are at (as they say):

The sharing model for fraction does tell us, through unintuitive technical reasoning, what the product of two fractions should be:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

(This answer comes from looking at the answers to division problems.) But this technical argument has no intuition or "good feel" about it.

It is a lovely coincidence that the "fraction of a fraction" computations have this same answer:

"$\frac{a}{b}$ of $\frac{c}{d}$" equals $\frac{ac}{bd}$

This approach has plenty of intuition attached to it, but there is no obvious reason to link the word "of" with the mathematical action of "multiplication." Nonetheless, people do, and just proclaim:

Define $\frac{a}{b} \times \frac{c}{d}$ to mean "a beeths of c deeths."

They are technically not wrong in doing this, but it makes it very difficult to answer the question "why" with something other than "It just works." This is not very satisfying.

For those who don't mind vacillating between "fractions as lengths on the number line" and "fractions as portions of pie," then the area model does too seem to suggest that the product of fractions should be defined as $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. Most people don't seem to notice that they keep switching models when thinking with the number-line, and so most people do not object to this.

In any case, we now have solid, technically correct means know to handle the arithmetic of fractions.

TO SUMMARIZE:

To add or subtract fractions, work with a common denominator. (Making use of the Key Fraction Property.)

To divide fractions, follow common sense with the Key Fraction Property.

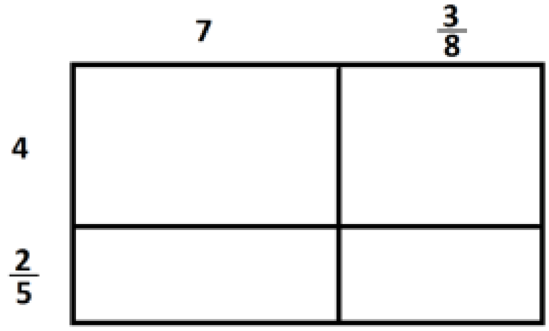
To multiply fractions, use $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ (knowing that this happens to match the "fraction of a fraction" idea and the area model from the number-line approach).

MULTIPLYING MIXED NUMBERS

Knowing that the product of fractions is consistent with the area model for multiplication we can use it to guide us in our thinking.

Example: *Compute* $4\frac{2}{5}\times 7\frac{3}{8}$.

Answer: This is the area of a rectangle, which naturally divides into four pieces:



We see that

$$\begin{aligned} 4\frac{2}{5}\times 7\frac{3}{8} &= 4\times 7 \ + \ 4\times\frac{3}{8} \ + \ \frac{2}{5}\times 7 \ + \ \frac{2}{5}\times\frac{3}{8} \\ &= 48+\frac{12}{8}+\frac{14}{5}+\frac{6}{40} \\ &= 48+1+\frac{1}{2}+2+\frac{4}{5}+\frac{3}{20} \\ &= 51+\frac{10+16+3}{20} \\ &= 51+\frac{29}{20} \\ &= 52\frac{9}{20}. \end{aligned}$$

FRACTIONS WITH NEGATIVE NUMERATORS AND DENOMINATORS

Mathematically “-2” represents the opposite of “2”, in the sense that adding 2 and -2 together gives zero. (Those familiar with my EXPLODING DOTS course at www.gdaymath.org know that I like to work with *dots* and *anti-dots*.)

The usual rules of arithmetic also allow us to think of -2 as $(-1)\times 2$ if we prefer.

Suppose now we can extend our work with fractions to include negative numbers as numerators and denominators. (Sharing anti-pie among anti-boys?)

Here’s a question:

Are $\frac{-3}{5}$ *and* $\frac{3}{-5}$ *and* $-\frac{3}{5}$ *the same fraction or are they all different as numbers?*

It is hard to answer this question with our sharing model or in any of our models. (Is $\frac{-3}{5}$, the result of sharing of three anti-pies to five boys, the same as the result of $\frac{3}{-5}$, sharing of three pies to five anti boys? And are these answers both three-fifths of an anti-pie?)

But if believe that the Key Fraction Property should hold for all types of numbers, including negative ones, then we have:

$$\frac{-3}{5}=\frac{-3\times(-1)}{5\times(-1)}=\frac{3}{-5}.$$

And we also have by our First Multiplication Belief 4 that:

$$-\frac{3}{5}=(-1)\times\frac{3}{5}=\frac{(-1)\times 3}{5}=\frac{-3}{5}\quad.$$

This shows all three quantities should be deemed the same number.

People call writing $\frac{-a}{b}$ as $-\frac{a}{b}$, and writing $\frac{a}{-b}$ as $-\frac{a}{b}$, as “pulling out a negative sign.”

Question 18:

a) What is $\frac{-a}{-b}$?

b) What is $\frac{-8}{9} \times \frac{2}{-5}$?

DIVIDING BY ZERO? A DENOMINATOR OF ZERO?

Our First Multiplication Belief says that $b \times \frac{a}{b} = a$. This provides a check to see if a fraction calculation is correct.

For example, suppose I am having trouble computing $\frac{20}{4}$. I think the answer is 3.

The First Multiplication Belief says that $4 \times \frac{20}{4} = 20$ and so my guess of three for $\frac{20}{4}$ is wrong as 4×3 is not 20.

In general, one checks whether or not a division problem is correct by performing multiplication. For example:

$\frac{6}{2} = 3$ is correct because 2 times 3 is indeed 6.

$\frac{20}{4} = 5$ is correct because 4 times 5 is indeed 20.

$\frac{83}{9} = 11$ is not correct because 9 times 11 is not 83.

$\frac{18}{0.1} = 180$ is correct because 0.1 times 180 is indeed 18.

Thinking Question 19:

a) Cyril says that $\frac{5}{0}$ equals 2. Why is he not correct?

b) Ethel says that $\frac{5}{0}$ equals 17. Why is she not correct?

c) Wonhi says that $\frac{5}{0}$ equals 887231243. Why is he not correct?

d) Duane says that there is no answer to $\frac{5}{0}$. Explain why he is correct.

Thinking Question 20:

Cyril says that $\frac{0}{0}$ equals 2.

Ethel says that $\frac{0}{0}$ equals 17.

Wonhi says that $\frac{0}{0}$ equals 887231243.

Why do they each believe that they are correct? What might Duane say here?

To answer these questions …

Notice that if $\frac{5}{0}=2$, as Cyril says, then we should have that 2 times 0 is 5, according to the check. This is not correct. In fact, the check shows that there is no number x for which $\frac{5}{0}=x$.

On the other hand, Cyril says that $\frac{0}{0}=2$ and he believes he is correct because it passes the check: 2 times 0 is indeed zero. But so too do $\frac{0}{0}=17$ and $\frac{0}{0}=887231243$ pass the check! In fact, $\frac{0}{0}=x$ passes the check for <u>any</u> number x.

The trouble with $\frac{a}{0}$ (with a not zero) is that there are no meaningful values to assign to it, and the trouble with $\frac{0}{0}$ is that there are too many possible values to give it!

In general, most people would say that dividing by zero is “undefined.” There is no means to give either an answer that is consistent with the arithmetic. The First Multiplication Belief suggests then that we should never allow the denominator of a fraction to be zero.

MULTIPLYING AND DIVIDING BY NUMBERS BIGGER AND SMALLER THAN ONE

People say that multiplying a quantity by a number bigger than one gives a bigger answer bigger. Is this true?

For instance, $\frac{5}{4}$ represents more than one pie. Does multiplying 100, for example, by $\frac{5}{4}$ give an answer bigger than 100?

Well, yes:

$$\frac{5}{4}\times 100=\frac{500}{4}=125\,.$$

Does multiplying any number, let's call it X, by $\frac{5}{4}$ give an answer larger than X ?

The answer is yes, and here it is good to write $\frac{5}{4}$ as a mixed number, $1\frac{1}{4}$, to see why.

$$\begin{aligned}\frac{5}{4}\times X &= \left(1+\frac{1}{4}\right)X\\ &=1\cdot X+\frac{1}{4}\cdot X\\ &= X+\ \textit{more.}\end{aligned}$$

Yes, the answer is bigger than X .

Does multiplying a quantity by a number smaller than one give a smaller answer?

Consider $\frac{4}{5}$, for instance. This represents less than one pie. Does multiplying 100 by it give a smaller answer?

$$\frac{4}{5} \times 100 = \frac{400}{5} = 80 .$$

Yes!

Does multiplying any number X by $\frac{4}{5}$ give an answer smaller than X ?

The answer is yes but we need to write $\frac{4}{5}$ as a mixed number in an unusual way: $\frac{4}{5} = 1 - \frac{1}{5}$. (So this is the mixed number $1\frac{-1}{5}$?) This shows:

$$\begin{aligned} \frac{4}{5} \times X &= \left(1 - \frac{1}{5}\right) X \\ &= X - \frac{1}{5} \cdot X \\ &= \text{ smaller than } X . \end{aligned}$$

Now let's consider dividing a number by a quantity smaller than one. For example, will 100 divided by $\frac{4}{5}$ give an answer smaller or larger than 100? Let's see:

$$\frac{100}{\frac{4}{5}} = \frac{100 \times 5}{\frac{4}{5} \times 5} = \frac{500}{4} = 125$$

The answer is larger.

In general:

$$\frac{X}{\frac{4}{5}} = \frac{X \times 5}{\frac{4}{5} \times 5} = \frac{5X}{4} = \frac{5}{4} \times X$$

and we know that $\frac{5}{4} \times X$ will be larger than X .

Question 22: Show that dividing a number X by $\frac{5}{4}$ will give an answer smaller than X .

ALGEBRA CONNECTIONS (for those with high-school mathematics experience)

In an advanced algebra course students are often asked to work with complicated expressions of the following ilk:

$$\frac{\frac{1}{x}+1}{\frac{3}{x}}.$$

We can make it look friendlier by following exactly the same technique of our fraction work. In this example, let's multiply the numerator and denominator each by x. (Do you see why this is a good choice?) We obtain:

$$\frac{\left(\frac{1}{x}+1\right)\times x}{\left(\frac{3}{x}\right)\times x}=\frac{1+x}{3}$$

and $\frac{1+x}{3}$ is much less scary.

As another example, given:

$$\frac{\frac{1}{a}-\frac{1}{b}}{ab}$$

one might find it helpful to multiply the numerator and the denominator each by a and then each by b:

$$\frac{\left(\frac{1}{a}-\frac{1}{b}\right)\times a\times b}{ab\times a\times b}=\frac{b-a}{a^2b^2},$$

and for

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{(w+1)^2}+5}$$

it might be good to multiply top and bottom each by $(w+1)^2$:

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{(w+1)^2}+5}=\frac{1-2(w+1)^2}{1+5(w+1)^2}.$$

Question 21: Make each of the following expressions look less scary:

a) $\dfrac{2-\frac{1}{x}}{1+\frac{1}{x}}$

b) $\dfrac{\frac{1}{x+h}+5}{\frac{1}{x+h}}$

c) $\dfrac{1}{\frac{1}{a}+\frac{1}{b}}$

d) $\dfrac{\frac{1}{x+a}-\frac{1}{x}}{a}$

e) $\dfrac{1}{s^{-2}}$

THE TRUTH ABOUT FRACTIONS

We like to believe that there are numbers that deserve to be called "fractions." And we like to believe that these are numbers that lie at positions between the whole numbers on the number line.

We like to believe that these numbers can be added, subtracted, divided, and multiplied, that is, they are numbers in their own right and have their own arithmetic rules.

But these things called fractions are hard! They are slippery and seem to slip around, through and between the models we create to describe them. And no one model seems to pin them down utterly and completely.

Nonetheless, all models for them start with the idea that fractions fill a philosophical gap in our understanding of numbers, that fractions "complete" our picture of multiplication just as negative numbers complete our understanding of addition. To explain:

In the world of positive counting numbers we can the solve equations

$$x+5=7 \text{ and } 42+w=100 .$$

But the positive counting numbers fail to solve equations such as:

$$x+8=3 \text{ and } 42+q=20 .$$

With the invention of negative numbers, however, all equations of the form $x+a=b$ now have solutions.

So by expanding our number system from just counting numbers (positive whole numbers) to the set of all integers, we've filled an apparent deficiency: we can now solve all addition equations.

Wonderful! But we're not out of the woods.

In the world of integers, some multiplicative equations have solutions:

$$5x=20 \text{ and } 6q=-18 \text{ ,}$$

for example, do but many do not, such as:

$$7x=10 \text{ and } 3w=2 .$$

> So we expand our number system again so as to include all solutions to equations of this multiplicative type: $bx = a$ with a and b (non-zero) whole numbers. We call the expanded system of the set of rational numbers – the whole numbers with fractions.

And just to continue the story …

> It was a shock to mankind that some equations of the form $x^n = a$ can be solved in the world of rational numbers, such as, $x^2 = 49$ and $w^3 = 8$, but not all such equations: $x^2 = 2$, for example. (It is not at all obvious that $\sqrt{2}$ fails to be a fraction – hence the shock to mankind!)
>
> So the system of rationals was also expanded to the set of all algebraic numbers so as to incorporate solutions to equations of these type too. By working with the decimal expansions of numbers, this system was expanded some more to the set of all infinite decimals, the real numbers.
>
> But the story doesn't end there, as some equations still fail to have solutions in the set of reals, the most famous being $x^2 = -1$.
>
> With the invention of the complex numbers, we now have a system of numbers for which each and every polynomial equation $ax + b = c$, $ax^2 + bx + c = d$, $ax^3 + bx^2 + cx + d = e$, and so on, is certain to have solutions. (This is an astounding feature of the complex numbers: they represent the end of this expanding story. Phew!)

Back to fractions:

The truth about fractions is that these quantities are actually an abstract concept designed to solve an abstract problem: make sure all multiplication problems have solutions. Thus mathematicians take the following approach to fractions:

> *Given any equation of the form $ax = b$ with a and b whole numbers, we'll say there exists a unique number which solves the equation. We denote this number: $\frac{a}{b}$.*

Notice right away we have a belief: $\frac{a}{b}$ is the solution to the equation $bx = a$ and so we have:

$$\boxed{(1)\ b \times \frac{a}{b} = a\ .}$$

One problem:

The equation $bx = a$ is equivalent to the equation $2bx = 2a$. So this means the fraction $\frac{2a}{2b}$ must be deemed the same as the fraction $\frac{a}{b}$. Similarly the fraction $\frac{3a}{3b}$ must equal $\frac{a}{b}$ as it is the solution to $3bx = 3a$, which is just the equation $bx = a$ in disguise. And so on. That is, we need to assume that out Key Fraction Property holds:

$$\boxed{\text{(2) } \frac{a\lambda}{b\lambda} = \frac{a}{b}.}$$

Obviously the equation $x = a$ has solution a so it follows that:

$$\boxed{\text{(3) } \frac{a}{1} = a.}$$

And the equation $ax = a$ has solution 1:

$$\boxed{\text{(4) } \frac{a}{a} = 1.}$$

Going further…

I claim that $\frac{e}{b} + \frac{f}{b}$ is a solution to the equation $bx = (e + f)$.

Let's check:

$$b\left(\frac{e}{b} + \frac{f}{b}\right) = b \times \frac{e}{b} + b \times \frac{f}{b} \qquad \text{by the usual distributive property}$$

$$= e + f \qquad \text{by (1) above.}$$

This means that $\frac{e}{b} + \frac{f}{b}$ is $\frac{e+f}{b}$. We have just proven:

$$\boxed{\text{(5) } \frac{e}{b} + \frac{f}{b} = \frac{e+f}{b}.}$$

So we know now how to add fractions with a common denominator. The Key Fraction Property (2) then suggests a method for adding fractions with different denominators.

How do we divide fractions? What is $\frac{a}{b} \div \frac{c}{d}$? Let's assume the answer exists for the moment and call it x. Then:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = x .$$

This means we want:

$$\frac{a}{b} = x\frac{c}{d} .$$

Multiplying through by b and by d with the aid of (1) means x is the solution to the equation

$$ad = xbc .$$

The solution is, by definition, $\frac{ad}{bc}$.

This suggests we should define $\frac{a}{b} \div \frac{c}{d} = \frac{ab}{cd}$ (that is, just as the result of applying the Key Fraction Property to $\frac{a/b}{c/d}$ suggests) .

How should we define the product of two fractions $\frac{a}{b} \times \frac{c}{d}$? Let's assume the answer exists for the moment and call it x. Then:

$$\frac{a}{b} \cdot \frac{c}{d} = x .$$

Multiply by b and by d to rewrite this, using (1) to see we have the equation:

$$ac = xbd .$$

The solution is, by definition:

$$x = \frac{ac}{bd} .$$

IN SUMMARY:

Analyzing solutions to equations of the form $bx = a$ with a and b whole numbers (and we should declare b non-zero) suggests there are numbers, called fractions, of the form $\frac{a}{b}$, with the following properties:

Fractions come in equivalent forms: $\frac{a\lambda}{b\lambda} = \frac{a}{b}$ **.**

Fractions add (and subtract) by: $\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}$ **.**

Fractions multiply and divide by:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

Done!

This is not very intuitive or exciting but it is the mathematical truth of what fractions are. One can't say anything more than that.

And this why fractions are so hard for students.

We cannot share the above mathematical story with beginning students: it is not at all pedagogically appropriate. So we must present models for these objects, and change models as our sophistication of work with fractions changes.

Each model is inadequate to some degree, and we and our students can sense that. But we don't have the means to do anything about it – we never, in the end, share with students what a fraction actually is:

> *A fraction is number we like to believe exists that solves an equation of the form* $bx = a$ *. These are the equations that usually arise in division or sharing problems.*

To make matters worse:

> *We like to believe that solutions to these equations are unique. This forces to say that many different looking fractions are actually equivalent:*

$$\frac{a}{b} = \frac{2a}{2b} = \frac{3a}{3b} = \cdots = \frac{-a}{-b} = \frac{-2a}{-2b} = \cdots = \frac{0.1a}{0.1b} = \frac{\sqrt{2}a}{\sqrt{2}b} = \cdots$$

And for the savvy student, we see that there is a potential problem with our definitions of addition, subtraction, division, and multiplication that needs to be attended to:

Question 22:
a) In our definition of addition for fractions we wrote:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

If we worked with an equivalent version of the fraction $\frac{a}{b}$ and an equivalent version of the fraction $\frac{c}{d}$, and applied this addition definition to those equivalent fractions, is the result sure to be a fraction equivalent to $\frac{ad+bc}{bd}$?

b) Similarly, are our definitions for multiplication and division also “well defined”?

Yeesh! Fractions are hard!

EXTRA: A BRIEF INTRODUCTION TO EGYPTIAN FRACTIONS

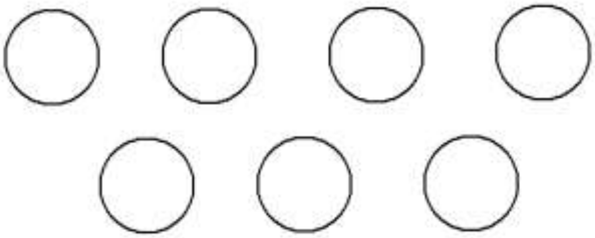
(See *THINKING MATHEMATICS! Vol 1: Arithmetic = The Gateway to All* available at www.lulu.com for more.)

Scholars of ancient Egypt (ca. 3000 BCE) were very practical in their approaches to mathematics and always sort answers to problems that would be of most convenience to the people involved. This led them to a curious approach to thinking about fractions.

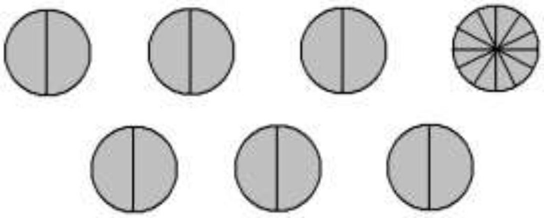
Consider the problem: *Share 7 pies among 12 boys.*

Of course, given our model for fractions, each boy is to receive the quantity "$\frac{7}{12}$" of pie. This answer has little intuitive feel.

But suppose we took this task as a very practical problem. Here are the seven pies:

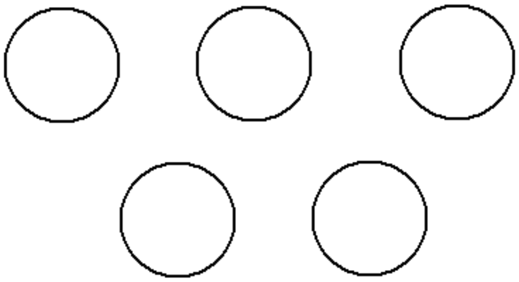


Is it possible to give each of the boys a whole pie? No. How about the next best thing – each boy half a pie? Yes! There are certainly 12 half pies to dole out. There is also one pie left over yet to be shared among the 12 boys. Divide this into twelfths and hand each boy an extra piece.



Thus each boy receives $\frac{1}{2}+\frac{1}{12}$ of a pie and it is indeed true that $\frac{7}{12}=\frac{1}{2}+\frac{1}{12}$.

Question 23:
a) How do you think the Egyptian's might have shared five pies among six girls?



b) How might they have shared 12 pies among 7 students?

The Egyptians insisted on writing all their fractions as sums of fraction with numerators equal to 1. For example:

$$\frac{3}{10} \text{ was written as } \frac{1}{4}+\frac{1}{20}$$

$$\frac{5}{7} \text{ was written as } \frac{1}{2}+\frac{1}{5}+\frac{1}{70}$$

That is, to share 3 pies among 10 students, the Egyptians said to give each student one quarter of a pie and one twentieth of a pie.

To share 5 pies among 7 students, the Egyptians suggested giving our half a pie, and one fifth of a pie, and one seventieth of a pie to each student.

Question 24: It is true that $\frac{4}{13}=\frac{1}{4}+\frac{1}{18}+\frac{1}{468}$. What does this say about how the Egyptians would have shared 4 pies among 13 girls?

Curiously, the Egyptians did not like to repeat fractions. Although it is obviously true that:

$$\frac{2}{5}=\frac{1}{5}+\frac{1}{5}$$

the Egyptians really did think it better to give each person receiving pie piece as large as possible, and so preferred the answer:

$$\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$$

(even though it meant giving out a tiny piece of pie with that bigger piece).

Question 25 Consider the fraction $\frac{2}{11}$.

a) Show that $\frac{1}{5}$ is bigger than $\frac{2}{11}$.

b) Show that $\frac{1}{6}$ is smaller than $\frac{2}{11}$.

c) Work out $\frac{2}{11}-\frac{1}{6}$.

Use c) to write $\frac{2}{11}$ the Egyptian way.

Question 26 Consider the fraction $\frac{2}{7}$.

a) What is the biggest fraction $\frac{1}{N}$ that is still smaller than $\frac{2}{7}$?

b) Write $\frac{2}{7}$ the Egyptian way.

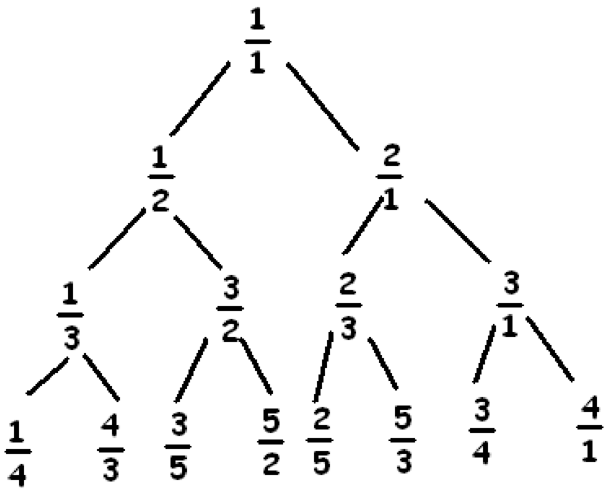
Question 27: CHALLENGE

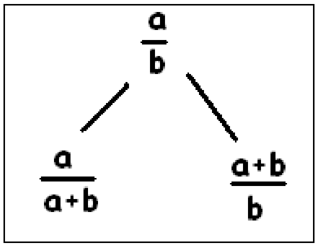
a) Write $\frac{17}{20}$ the Egyptian way.

b) Write $\frac{3}{7}$ the Egyptian way.

EXTRA: A CURIOUS FRACTION TREE

Here is something fun to think about. Consider the following "fraction tree:"





Do you see how it works? Do you see that each fraction has two "children"? The left child is always a number smaller than 1 and the right child is always a number larger than 1.

Do you see how the box to the upper right gives the method for computing the two children of the fraction?

a) Continue the drawing the fraction tree for another two rows.

b) Explain why the fraction $\frac{13}{20}$ will eventually appear in the tree. (It might be easier to figure out what $\frac{13}{20}$'s parent is by first noticing that $\frac{13}{20}$ is a "left child." What is its grandparent? What is its great grand parent?)

c) Explain why the fraction $\frac{13}{20}$ cannot appear twice in the tree.

d) Will the fraction $\frac{456}{777}$ eventually appear in the tree? Could it appear twice?