CURRICULUM INSPIRATIONS

TIDBITS FOR THE CLASSROOM INSPIRED BY MAA AMERICAN MATHEMATICS COMPETITIONS

ESSAY NUMBER: 6

Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty. (From amc.maa.org.)

This is a sixth essay to prove this right!

Each time we take a question from the MAA AMC and model a classroom discussion that you could lead. We tie the concepts at hand directly to the curriculum (you are not wasting precious class time!) and model the true experience of doing mathematics.

Success, even on mandated state tests, requires students to rely on a sense of self confidence: the confidence to acknowledge one's emotions and to calm them down, the confidence to pause over ideas and come to educated conclusions, the confidence to believe in one's own wits to navigate different, or even unfamiliar, terrain, and the confidence to choose understanding over impulsive rote doing.

Success and joy in science, business and in life doesn't come from programmed responses to pre-set situations. They come from agile and adaptive thinking coupled with reflection, assessment, and further adaptation.

Each MAA AMC question, as a stand-alone item, can teach confidence and mindful agility. Here's yet another concrete example showing how!

Our tidbit today is query 23 from the 2004 AMC 8 experience. It is considered quite an exploration for 8th graders. As we shall see, it leads to good thought and activity for 9th to 12th graders as well!

COMMON CORE STATE STANDARDS and PRACTICES:

The interplay between verbal and geometric descriptions of functional relationships is a key component of the common core state standards. The MAA AMC topic of this essay addresses the Grade 8 Standard:

8.F: *Define, evaluate and compare functions*

and the high-school standards:

F-IF: *Analyze functions using different representations.* **F-IF-7:** *Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.* **F-IF-9:** *Compare properties of two functions each represented in a different way.* **F-BF-1:** *Write a function that describes a relationship between quantities* and **G-MG:** *Apply geometric concepts in modeling situations.*

In addition we shall discuss velocity, distance and the distance formula, and parametric equations. We can also segue into circular motion and the circle functions of trigonometry.

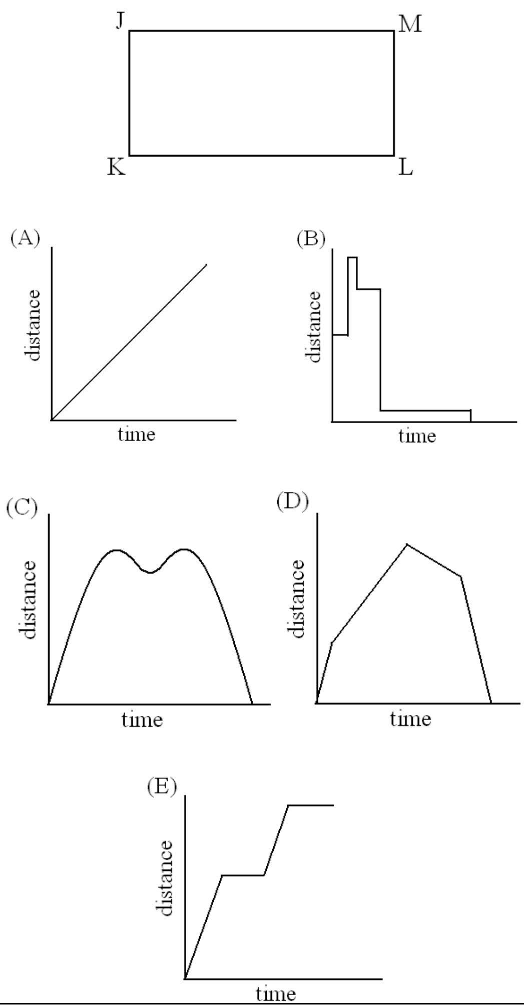
We also model practice standards:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
7. Look for and make use of structure.

Whoa!

Here is the question:

Tess runs counterclockwise around rectangular block $JKLM$. She lives at corner J. Which graph could represent her straight-line distance from home?



The first step in any problem-solving experience is to:

STEP 1: `Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.`

You, as an educator having just read this question, are most likely having a different emotional response to it than your students will. You might be thinking:

Oh heavens, there is a lot here for young students to parse. They need to know how to read and interpret a word question AND they need to know which direction is counterclockwise AND they need to figure out what "straight-line distance" could mean (and so do I!) AND they need to process and translate personal understanding into visual distance/time graphs AND they need to....

How can I possibly "teach" kids how to do all this?

Your students reading this might be thinking: *Huh?* And when pressed further they might dub the problem as scary.

After sharing the question with your class, do have students verbalize all their initial thoughts, reactions and emotions as a general class discussion, I suggest that you next describe exactly what you are feeling too as a teacher! Express your professional qualms using the same language and phrases as you would with a colleague. Give your students the experience of seeing what educators go through and thereby break down the idea that someone is just meant to somehow know math and always know what to do right off the bat. Share with them your personal experience. (Try it!)

Comment: Many teachers feel that it is their job to be the expert at all times. So what then is the best way to be an "expert" when teaching students how to "nut their way" through problems, to flail, to try again, to make mistakes? By always being instantly perfect can one model and support the true organic back-and-forth of doing human, original thinking?

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After your students have shared their initial feelings and you your perceived challenges and concerns, go to:

STEP 2: `Understand the question. Understand the different components of the question.`

Have a class discussion of mutual exploration. Include yourself as one also seeking personal clarity.

What is the question about?
What is Tess doing?

Well… Tess is running around this rectangle.

More detail:
She is starting at corner J.

(Actually… Do we know that? All we know is that she lives at corner J.)

More detail:
She is running counterclockwise.

What does counterclockwise mean?

(A quick class conversation will sort this out.)

What does the question want us to do?

Dunno.

What do you mean by "dunno"?
It's asking for something weird.

What exactly?
Her "straight-line distance from home."

Do we know what that means?
Nope.

Do we know anything about the answers given to us (even if we don't understand the question!)?

Those pictures are definitely scary!

So we have pinned down two points of focus:

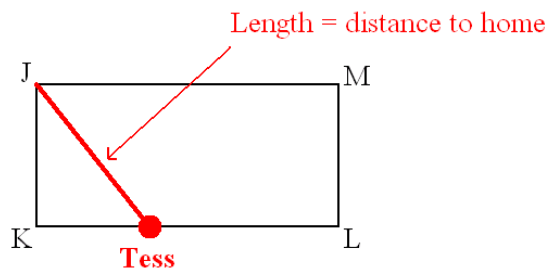
1. ***What does "straight-line distance from home" mean?***
2. ***What is a distance/time graph and how do you interpret one?***

Straight-Line Distance:
One lovely feature of this question is that it uses a term that is not standard in the vocabulary of math words and terms. We are simply being asked (forced!) to make an intelligent guess as to what the author means by some language. (How fabulous!) Share this observation with your students.

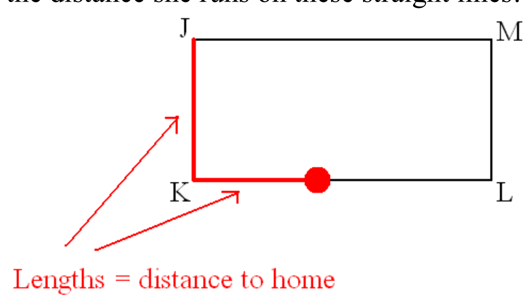
So, what could "straight-line distance" mean?

During a class discussion, students will likely come up with two possibilities. (And knowing youngsters, I am sure they will come up with more!)

<u>Possible Meaning One</u>: As Tess runs around the block her "straight-line distance" from home would be the length of a straight line that connects her to home.



<u>Possible Meaning Two</u>: But Tess <u>is</u> running on straight lines! Her straight-line distance is the distance she runs on these straight lines!



Which one do you think the author means?

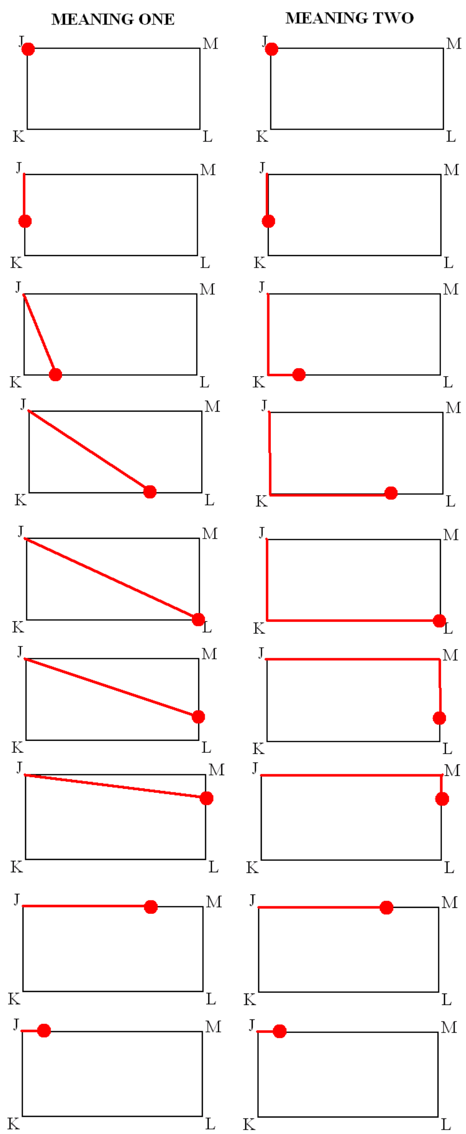
This could lead to an interesting class discussion!

HOWEVER, without really having settled anything, we can employ a standard problem-solving strategy for test questions…

IN A MULTIPLE-CHOICE QUESTION:
TRY TO ELIMATE ONE OR MORE OF THE SUGGESTED OPTIONS?

Best of all… try to eliminate all but one of the given options. This will reveal what must be the answer!

Let's look at how Tess's distance from home changes as she goes around the block – for each of our two meanings.



(Notice in the second interpretation how Tess's shortest path to home changes at the point L.)

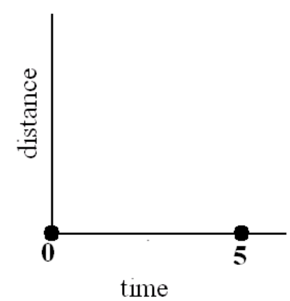
In <u>both</u> interpretations, Tess starts at home (ooh! Did we decide this?) at zero distance. Her distance from home then increases, reaches a maximum distance at the single point L, and then decreases back down to zero.

So at her starting time her distance is zero, and at a later time her distance is again zero. Even though we haven't addressed what is a distance/time graph is we can ask:

Which of options (A) through (E) seem to have this property?

The class will invariably settle on options (C) and (D). And, without having made it a formal "teaching discussion," some understanding of what a distance/time graph means will naturally emerge as part of this conversation.

Comment: If a teacherly nudge is needed, you might want to point out that the horizontal axis is time. If Tess starts at time 0 minutes, and then takes, say, 5 minutes to run all the way round, then we'd expect a "distance/time graph" to show a distance of zero at time 0 and time 5.



Next:

Which of the two graphs(C) or (D) is correct? OR, maybe it is better to ask, which is not correct?

Students will no doubt say that graph (C) is wrong because it has two places of biggest distance. Tess's motion (in both interpretations) has only one.

This leaves (D) as the only possible answer to the question!

A WONDERFUL OPPORTUNITY:
So … have we experienced a success? We have the answer: job done! So all is good and we feel satisfied?

At this point I personally feel dissatisfied! I may have a bit of a clearer understanding of what a distance/time graph is, and I may have sorted out in my mind once and for all which direction is counterclockwise, but I don't feel I have actually answered anything in this question. I haven't settled the issue of what "straight-line distance" might mean, I haven't at all thought about graph (D) directly and if it could be right. (I've simply only eliminated all other choices.) And I don't feel like I have any sense of owning and truly understanding this question!

I don't like this feeling of squeaking my way through a situation by the skin of my teeth. I got exactly what was asked of me done, sure, but not one whit more. Who wants to go through life only doing the minimum of what is asked of them? I want to be a leader in thinking, innovation, and success!

This question is wonderful as **NONE of the graphs shown are the true graphs of Tess's motion and her straight-line distance from home (for either interpretation)!** Well, that is not quite right. **ALL the graphs shown could be a graph of her motion and straight-line distance** if we question our assumptions! (Confused?) Who said she is starting from home? Is she running at uniform speed? Does she pause to catch her breath as she runs? Is the graph meant to show her complete journey around the block or perhaps only some part of it? Could she change direction for part of her run? Could this be happening in the 33rd century and Tess teleports to different points along her journey?

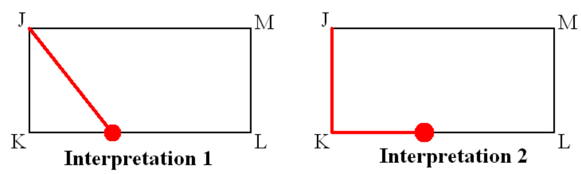
Suddenly this question opens up to reveal of whole universe of wonderful ideas and explorations. Now I am excited and eager to really learn from this MAA AMC problem!

Truly Understanding Distance/Time Graphs:

A super activity for students would be to take each answer offered in MAA AMC question 23 and ask: *Could this represent Tess's motion in some way?*

Here's what comes to my mind when thinking through this. Come to your own ideas too. Share one or two with them with your students to get the conversation started, but let them gradually take the reins and come up with their own valid possibilities.

Remember: We still have two interpretations of "straight-line distance" at play.



[Your students, by now, may have decided that the author meant the first interpretation. Most mathematicians reading question 23 would have assumed this too.]

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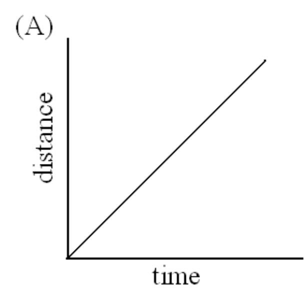

This graph starts with a distance 0 and shows that her distance from home increases at a uniform rate for a while. Here Tess could be…

<u>In interpretation one</u>: …running the first part of her journey from J to K at a uniform speed.

<u>In interpretation two</u>: …running from J to L at uniform speed. (Do you see why?)

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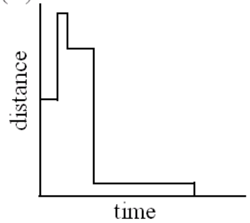
(B)



Maybe Tess starts at point K and stands there for a while. Then she instantly teleports to point L, pauses, instantly teleports to point M, pauses, teleports to a point right near home (but not quite), pauses for quite a while, and then instantly teleports home.

[Okay .. .this is far-fetched! The question did say "run around the block" after all. But it is a good exercise to try to make this answer work nonetheless.]

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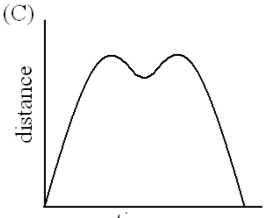
(C)



Maybe Tess runs to point L, varying her speed in some way, runs back for a spell and then returns to L (OR runs forward from L and then returns to L !!) and then finally carries on to run home.

Question: *Does this scenario work for both interpretations of "straight-line distance"?*

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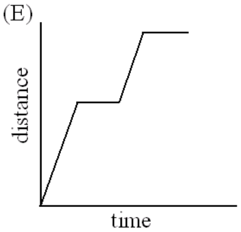
(E)



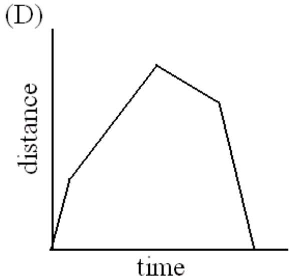
Maybe Tess starts at J, runs at a uniform speed for a while and pauses halfway to K, then runs to K at the same speed, and then pauses again. (This works for both interpretations of "straight-line distance.")

THE "CORRECT" ANSWER (D):
At this point it seems that we can make most any reasonable picture of a distance/time graph be "correct" if we have Tess change her speed as she runs, or pause here and there, or run backwards, and the like.

Question: *How "unreasonable" would a graph have to be in order for it not to mimic any kind of possible motion?*

We can certainly make option (D) work as a valid graph of motion.

(D)



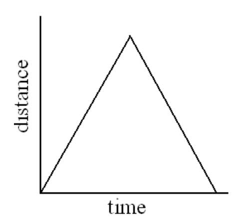
But one might question the linear sections of this graph. This linearity "feels" right, but is it?

Let's go with the assumptions the author of the question most likely wanted to us to make:

- **Tess starts her run at point J.**
- **She runs one full circuit of the block at a uniform speed.**

Is graph (D) actually correct in this circumstance?

Under the second interpretation of "straight-line distance" can you see that distance/time graph will be a symmetrical triangular graph?

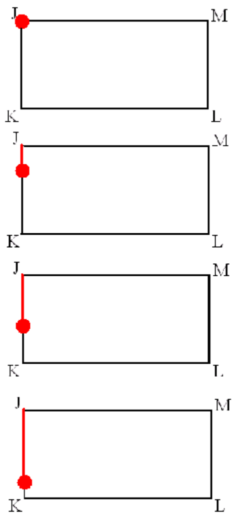


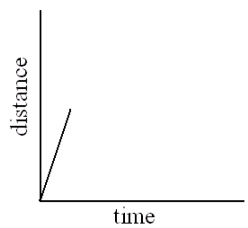
As this graph wasn't offered as an option, we might guess the author of question 23 did mean the first interpretation of "straight-line distance" after all.

We now have an interesting question:

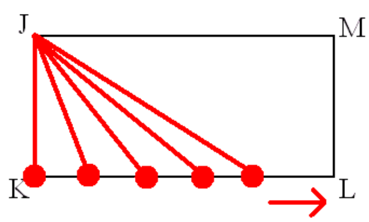
If Tess indeed starts running from point J at a uniform speed, how does the length of the straight line connecting her position to J change over time?

As Tess runs from J to K, we can see that her straight-line distance increases at a uniform rate:





As Tess runs from K to L matters are less clear. *Is the length of the red line increasing in a uniform manner?*



Student Modeling Project:
Using geometry software (or even pencil and paper), draw a point P on the screen and a line below it. Measure the distance from P to points at regular intervals along the line. (Why regularly spaced points?)

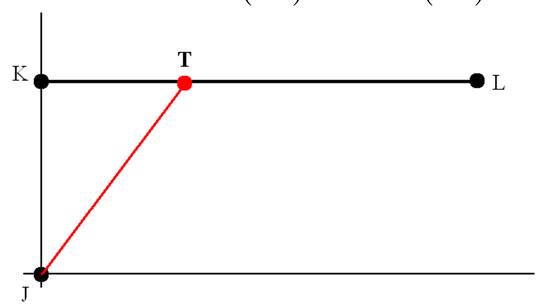
Draw a graph of your distances. Is it a straight line graph?

Student Pre-Calculus Project:
If your students have studied parametric equations, they can write an analytic formula for the distance between a fixed point J and a point T (for Tess) moving along a straight line.

[For a short and completely accessible video on parametric equations, go to www.jamestanton.com/?p=1108.]

Just to give a quick sense of the work you and your students could do along these lines…

Let's assume that J is at the origin of the Cartesian plane, and let's turn things around a bit and have $K=(0,1)$ and $L=(2,1)$.



Let's also assume that Tess moves from K to L in one unit of time.

Then Tess's x- and y-coordinates at any time t are given by:

$$x = 2t$$
$$y = 1$$

[Check: When $t=0$ she is at $(0,1)$, yep, that's K; and when $t=1$ she is at $(2,1)$, which is L.] Her distance from the origin at time t is $\sqrt{(x-0)^2+(y-0)^2}=\sqrt{4t^2+1}$.

a) Using software, plot a graph of $\sqrt{4t^2+1}$ for $0 \le t \le 1$. Is it a straight line graph?

b) Go back to the original problem and place the rectangle $JKLM$ on a Cartesian plane. Choose coordinates for each of the points J, K, L and M (make sure it is a rectangle) and write parametric equations for each portion of Tess's journey. Plot the true straight-line distance/time graph as she goes around the rectangle once.

We will see that option (D) is not correct!

But notice that the author of the question cleverly covered him/herself by only asking which graph could be correct. At first analysis, (D) is indeed the only possibly correct option.

Do sections of the graph of $\sqrt{4t^2+1}$ look straight? Is option (D) given in the original question, with straight line sections, close to being a correct solution?

We have a springboard now for many good curriculum related questions!

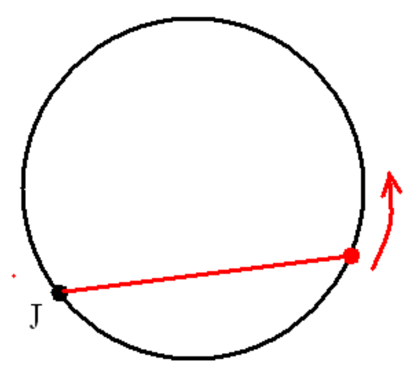
c) How would the distance/time graph change if Tess went around the rectangle twice? Three times?

d) How would the graph change if she went around the rectangle in a clockwise direction instead? (Something fun … look up the words *widdershins* and *deasil*.)

e) How would the graph change if she doubled her speed?

f) How would the graph change if she paused for one moment at each corner?

g) Suppose Tess went around a circle, starting at a point J on that circle. What does the distance/time graph look like in this case?



[And now let's study the x - and y - coordinates of circular motion and discover trigonometry!]

Have your students invent even more questions—and answer them too!