Curriculum Burst 25: Intersecting Tetrahedra

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> Two distinct regular tetrahedra have all their vertices among the vertices of the same unit cube. What is the volume of the region formed by the intersection of the tetrahedral?

SOURCE: This is question # 24 from the 2010 MAA AMC 10a Competition.

QUICK STATS:

MAA AMC GRADE LEVEL
This question is appropriate for the 10th and 12th grade levels.

MATHEMATICAL TOPICS
Geometry: Solids; Effect of Scale on Volume; Symmetry.

COMMON CORE STATE STANDARDS

G-GMD.4:   Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.
8.G.8:   Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

MATHEMATICAL PRACTICE STANDARDS
MP1   Make sense of problems and persevere in solving them.
MP2   Reason abstractly and quantitatively.
MP3   Construct viable arguments and critique the reasoning of others.
MP7   Look for and make use of structure.

PROBLEM SOLVING STRATEGIES
ESSAY 4:   **DRAW A PICTURE**
ESSAY 9:   **AVOID HARD WORK!**

# THE PROBLEM-SOLVING PROCESS:

The most important step is the first:

> **STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

My first reaction is … panic!

I know what a regular tetrahedron is (a solid with four congruent equilateral-triangular faces) and I know what a cube is. But there are tetrahedra sitting in cubes? In fact …two? I can't even see one.

Deep breath.

Obviously we want to draw pictures for this question. Here's a cube and here's a tetrahedron.
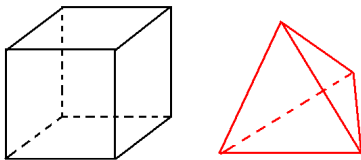
Two copies of the tetrahedron sit in the cube with all vertices (corners) lining up? The cube has 8 vertices and a tetrahedron has 4. Okay … here's how four vertices of a cube make a tetrahedron.
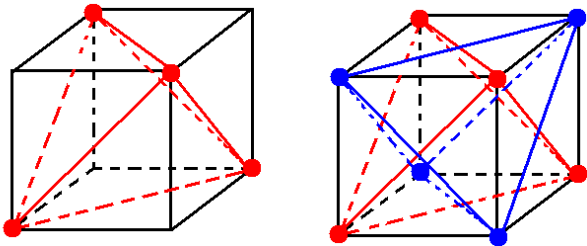
And, making the picture a real mess, we can draw in the second tetrahedron as well.

We want the volume of the intersection of these two tetrahedra. Yeesh! (I can't even see what shape the region of intersection is.)

> **AVOID HARD WORD!**

Just look at the red tetrahedron in the picture to the left. Each vertex is the apex of a region inside the tetrahedron but outside the region of intersection. By symmetry, each of these four "cap" regions, one for each vertex, must be identical in shape and volume. In fact, the same is true for the eight cap regions: the four for the red tetrahedron and the four for the blue.

What shape is each cap region?

Hmm. The edges of the red and blue tetrahedra cross at the centers of each face of the cube. Aah! Each "cap region" is its own tetrahedron with edges half the length.
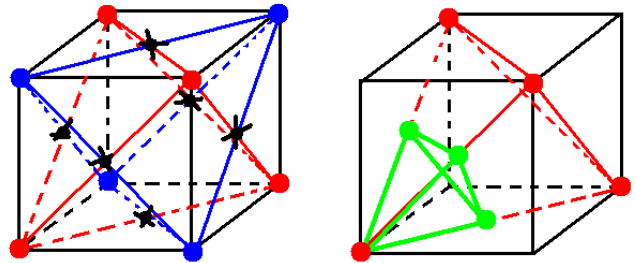

If the volume of one tetrahedron is V, then the volume of the cap is $\left(\dfrac{1}{2}\right)^3 V = \dfrac{V}{8}$. (Volume scales as the scale factor cubed.)

Ooh! Denote the volume of the intersection region as R. The intersection plus the four caps make the whole tetrahedron. We have: $R + 4 \times \dfrac{V}{8} = V$. Thus $R = \dfrac{V}{2}$. The intersection is half the volume of a single tetrahedron. This is cool!

What is the volume of the red tetrahedron? Again avoiding hard work we see that the complement of the tetrahedron consists of four triangular pyramids, each with base of area $1/2$ and height 1.
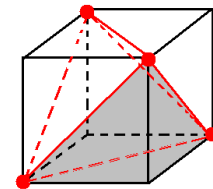
Thus $V = 1 - 4 \times \dfrac{1}{3} \cdot \dfrac{1}{2} \cdot 1 = \dfrac{1}{3}$. Consequently the area of the region of intersection we seek is $R = \dfrac{V}{2} = \dfrac{1}{6}$. Whoa!

> **Extension:** What is the shape of the region of intersection? (Use symmetry and pure thought to answer this!)

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