Sketch a graph of the function

$$p(x)=(x-50)^{37}(x-7)^{202}(x+3)^{99}(x+15)^{3003}(x+22)^{42}.$$

Response: Which *x*-values are interesting?

The values *x* = 50 *and* *x* = 10,000,000 seem interesting.
x = 7, *x* = -3, *x* = -15, *x* = -22; *x* = -10,000,000

TEACHING THE PROBLEM-SOLVING MINDSET

A Classroom Moment

MATHEMATICAL ASSOCIATION OF AMERICA

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Graphing Factored Polynomials

One 19th-century motivation for teaching the art of factoring of polynomials is that the task of sketching the graph of a polynomial in factored form is relatively straightforward—and even fun! It is a great exercise on relying of your wits and just "following your nose." Giving students the opportunity to practice common sense is a worthwhile 21st-centure enterprise. So let's make common-sense puzzling the goal of this particular pre-calculus topic.

Without any lead-in to this unit, just present your students a ghastly-looking problem. Ask the class to think about how one might go about sketching the graph of 3383-degree polynomial given in the image above.

$$p(x)=(x-50)^{37}(x-7)^{202}(x+3)^{99}$$
$$\times(x+15)^{3003}(x+22)^{42}$$

Step 1 in the problem-solving process is to

HAVE AN EMOTIONAL REACTION

and your students might indeed readily start this first step! The next move is to acknowledge one's trepidation, take a deep breath, and carry on to the next problem-solving step, which is to

DO SOMETHING!

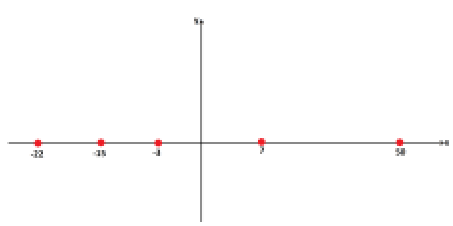
But what can be done? What possible start can one make?

The prompt you could provide here is the question:

Do there seem to be any values for x that might be interesting for this function?

Students will invariably say that $x=50, 7, -3, -15, -22$ are "interesting." They each give an output of zero.

Great! We have something to sketch on a graph. (Enjoy whatever success you have as they happed!)



Will the function be zero for any other inputs?

No! It is clear that $P(x)$ will be non-zero at $x=4$ and at $x=102$, and so on. So these five points are the only locations at which the graph touches the x-axis.

Do any other interesting x-values come to mind?

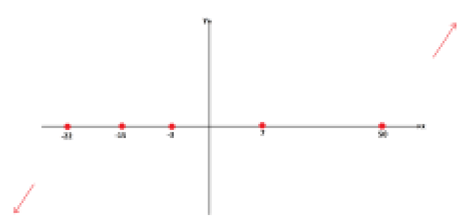
Most students will say no, but a mathematician might say otherwise.

GO TO EXTREMES!

How about a ludicrously large and positive x-value and a ludicrously large and negative one?

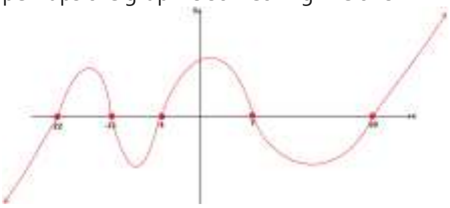
We readily see that $P(10,000,000)$ is large and positive, for instance and that $P(-10,000,000)$ is large and negative.

We can sketch this information too!



Is the shape of the graph coming into form?

Since we have identified the only places that the graph can touch the horizontal axis, perhaps the graph is something like this?



This is a beautifully tempting picture to draw. But is it right?

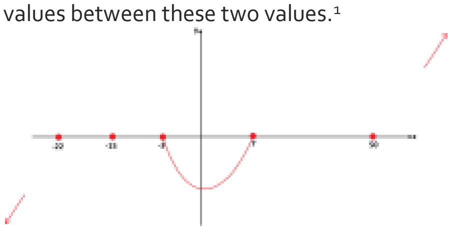
The graph we have suggests that $P(0)$ is a positive value? Is it?

Well

$P(0) = (-50)^{37}(-7)^{202}(3)^{99}(15)^{3003}(22)^{42}$

is negative. Hmm!

Actually, since $P(0)$ is negative, and the function is zero at $x = -3$ and $x = 7$ and nowhere else in-between, the graph of the polynomial must be negative for all x -values between these two values.[1]



Ahh! We can apply the same analysis for the remaining intervals between the zeros: the polynomial must be either entirely positive or entirely negative on each such interval. To find out which we need only check the sign of one output for any one input within each interval.

For example, along with

$$P(0) = negative$$

we have

$$P(-20) = (negative)^{37} \times (negative)^{202} \times (negative)^{99} \times (negative)^{3003} \times (positive)^{42} = negative$$

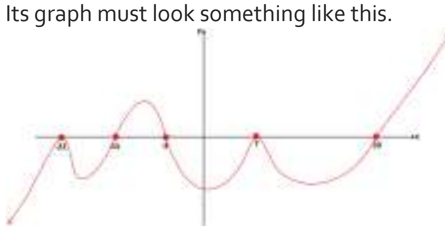
$$P(-10) = positive$$

and

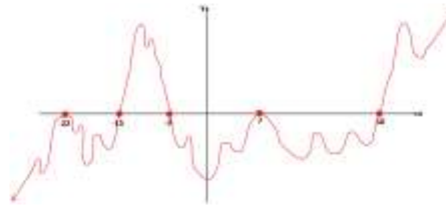
$$P(36) = negative .$$

We also already know that the polynomial takes positive values to the right of $x = 50$ and negative values to the left of $x = -22$.

Its graph must look something like this.



Well… we don't know this! All we currently know is that the polynomial is positive and negative within certain regions. Its graph could well look more like this.



This sort of picture has to be deemed a thoroughly acceptable answer to this task in a pre-calculus class. To ascertain how many "dips" and "humps" there are requires the tools of calculus and so is beyond scope of the typical course in which this unit sits.

Now having gone through this experience once, students are well set to try graphing all sorts of factored polynomials on their own. There are no rules to memorize ("odd exponents cross through, even exponents bounce"). Just ask students to do something! Start with some obvious x -values that seem to cry out, try some extreme ones too, and then follow your nose!
This is good 21st-century noodling for sure!

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[1] We are making the assumption here that the graph of a polynomial shall be a smooth continuous curve. Do we really know this?