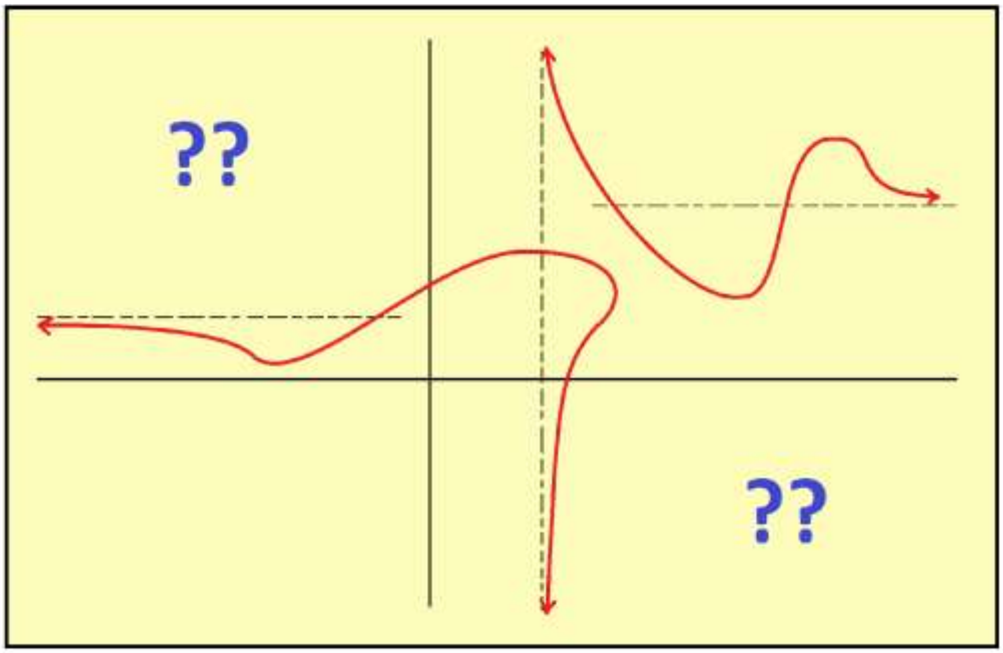


TEACHING THE PROBLEM-SOLVING MINDSET

A Classroom Moment

MATHEMATICAL ASSOCIATION OF AMERICA

BY JAMES TANTON

WWW.MAA.ORG/CI

Graphing Rational Functions II

A rational function is a function defined by an expression algebraically equivalent to one of the form $\frac{p(x)}{q(x)}$, with p and q each a polynomial. For example,

$$f(x) = \frac{2(x-1)(x-5)}{(x-2)(x-3)}$$

defines a rational function f. It is a standard practice of the upper high-school curriculum to have students sketch the graphs of rational functions.

One starts by observing that the x-values that make the value of the numerator or denominator of the rational expression vanish are interesting. Such values correspond to a location where the graph of the function crosses the x-axis, or where the graph has a hole, or where the graph has a vertical asymptote.

By **GOING TO EXTREMES**, that is, by considering extraordinarily large positive or negative values of x, one can identify horizontal or oblique asymptotes too.

After plotting the graphs of many rational functions it could be instructive to ask: *What is possible to see in the graph of a rational function?* Sharing the title picture of this essay with students can provoke all sorts of good questions.

Can the graph of a rational function cross a vertical asymptote?

Can the graph of a rational function cross a horizontal asymptote?

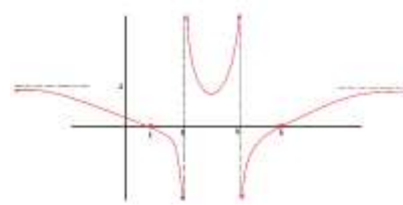
Must the value of a rational function have opposite signs just either side of a vertical asymptote?

Can a rational function have a horizontal asymptote of one height to the right of the graph and one of a different height to the left?

Must every rational function have an asymptote to the right and one to the left? One might argue, for instance, that the function $g(x) = x^2 + \frac{1}{x}$ has a parabolic asymptote. (And again, it is the same asymptote to the left and to the right. Hmm.)

What is a non-vertical asymptote anyway? Could one argue, for instance, that $h(x) = 2 + \frac{1}{x} + \frac{1}{x^2}$ has the hyperbola $y = 2 + \frac{1}{x}$ as an asymptote? Or must one say that the line $y = 2$ is an asymptote?

Even graphs of the standard textbook examples beg many questions. For example, students will likely draw the graph of the function f given earlier as follows.



Is the shape of the curve to the left of $x = 2$ that simple? Could the curve rise up above the asymptote and later settle towards it (perhaps even in an oscillatory fashion)?
Is the shape of the middle section of the graph correct? How low does it go? Is it, for certain. U-shaped? Could it be a "W" or have multiple dips and humps?
Is 2 in the range of this function?

Exploring such questions, and even deciding one doesn't have the tools to answer some of them, is certainly promoting a healthy problem-solving mindset.