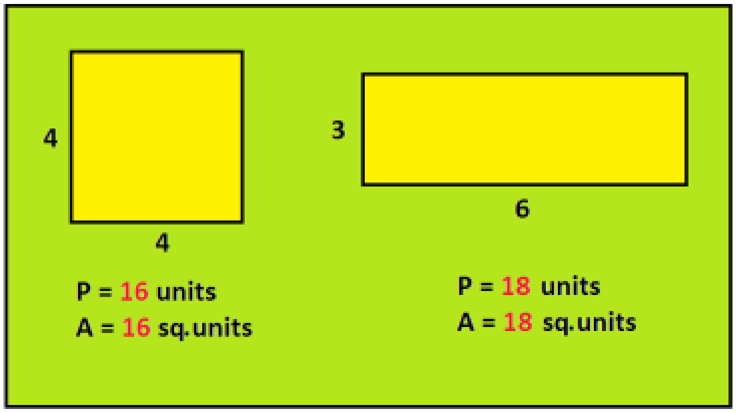

TEACHING THE PROBLEM-SOLVING MINDSET

A Classroom Moment

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The Remainder Theorem

Here's a classic puzzle:

The 4×4 square has the property that its area and its perimeter have the same numerical value. The 3×6 rectangle has this surprising property too. Find more examples of rectangles with integer side lengths having matching area and perimeter values.

This amusing puzzle serves as perfect segue into a discussion on the Remainder Theorem in high school polynomial algebra.

Recall, a polynomial is a function p whose outputs $p(x)$ can be expressed in the form

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

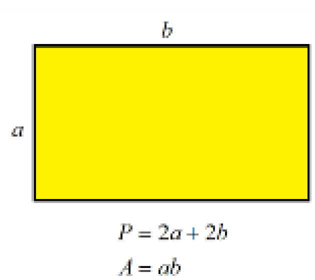
for some fixed values $a_0,\ a_1,\ldots,\ a_n$. Polynomials generalize the notion of base-ten place-value: we're now examining place-value in base x. (And we are allowed to set $x=10$ if we like!)

The Remainder Theorem says that if we divide a polynomial by a linear term, $x-h$, then the remainder (if any) will be just a single number and that number is $p(h)$, the value of the polynomial at $x=h$. Whoa!

The puzzle facilitates the discovery of this surprising fact. It encourages the right mindset for always being on the lookout for structure and simply doing your best just following your nose.

SOLVING THE PUZZLE

We seek an $a\times b$ rectangle with a and b positive integers satisfying $2a+2b=ab$.



What can we do with this equation?

DO SOMETHING!

It seems compelling to solve for one of the unknowns. Solving for b gives

$$b=\frac{2a}{a-2}.$$

It looks like that b "wants to be" a fraction. But we are meant to have b a whole number. Thus we must have $2a$ a multiple of $a-2$. That seems hard to orchestrate.

ENGAGE IN WISHFUL THIINKING

Can we make the numerator a multiple of the denominator? Is there an ideal numerator? If the numerator were $2a-4$, then we'd have $b=\frac{2a-4}{a-2}=2$, a good whole number!

Of course, we don't have this ideal situation. But let's take what we have and make it look like our dream answer as best we can.

$$b=\frac{2a}{a-2}=\frac{2a-4+4}{a-2}=2+\frac{4}{a-2}.$$

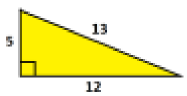
Ahh! Now we see that to obtain a whole number value for b we must have $a-2$ a factor of 4. It must be ±1, ±2, or ±4.

Checking these options in turn ...

* $a-2=1$ yields $a=3$, $b=6$, the 3×6 rectangle we already have.
* $a-2=2$ yields $a=4$, $b=4$, the square we already have.
* $a-2=4$ yields $a=6$, $b=3$, the 3×6 rectangle we already have.
* $a-2=-1$ yields $a=1$, $b=-2$, which is not permissible.
* $a-2=-2$ and $a-2=-4$ each yield non-permissible values for a.

Thus the 3×6 rectangle and 4×4 square are the only integer rectangles with this property! There are no more.

Challenge: The $5-12-13$ integer right triangle has perimeter 30 units and area 30 square units.



Find all other integer right-triangles with this "area equals perimeter" property.

Comment: If we scale a geometric figure of perimeter P and area A by a scale factor k, then the scaled figure has perimeter kP and area k^2A. Choosing $k=P/A$ yields a figure with perimeter matching area in numerical value. This shows that for any figure drawn on the page, there is a ruler of units for which the perimeter and area of the figure measured with that ruler have the same numerical value.

POLYNOMIAL DIVISION

Let's now leap to polynomial algebra.

What is $\dfrac{2x^2+7x+3}{x-1}$ *?*

Since the wishful thinking of the previous puzzle paid off, let's do it again!

But let's start by asking: Can we use our wishful thinking to "make" $2x^2+7x+3$ multiple of $x-1$?

Start with the $2x^2$ term. This is a multiple of x, namely, $2x(x)$, which is close to being a multiple of $x-1$. Let's arrange matters so that $2x(x-1)$, that is, $2x^2-2x$, appears.

$$\begin{aligned}2x^2+7x+3&=2x^2-2x+2x+7x+3\\&=2x(x-1)+9x+3\end{aligned}$$

The next term to consider is the $9x$. This is a multiple of x. Can we adjust matters so that $9(x-1)$, a multiple of $x-1$, appears instead? That is, can we make $9x-9$ appear?

$$\begin{aligned}2x^2+7x+3&=2x(x-1)+9x+3\\&=2x(x-1)+9(x-1)+9+3\\&=2x(x-1)+9(x-1)+12\end{aligned}$$

Okay. Now it is clear that

$$\frac{2x^2+7x+3}{x-1}=2x+9+\frac{12}{x-1}.$$

Practice: *Write* $p(x)=x^3-2x+1$ *as a multiple of* $x-5$, *if possible. (If not possible, get close!)*

Answer: The *Wishful Thinking* approach is fun!

The first term, x^3, is $x^2(x)$, a multiple of x. Let's make it a multiple of $x-5$. (Is my loose language okay?)

$$\begin{aligned}p(x)&=x^3-2x+1\\&=x^2(x-5)+5x^2-2x+1\end{aligned}$$

The next term, $5x^2$, is $5x(x)$, a multiple of x. Let's make it a multiple of $x-5$.

$$\begin{aligned}p(x)&=x^2(x-5)+5x^2-2x+1\\&=x^2(x-5)+5x(x-5)+23x+1\end{aligned}$$

Let's now adjust the $23x$ term.

$$\begin{aligned}p(x)&=x^2(x-5)+5x(x-5)+23x+1\\&=x^2(x-5)+5x(x-5)+23(x-5)+116\end{aligned}$$

And this is as far as we can go! We're close to being a multiple of $x-5$. We're off just by a single number, 116.

What is that number 116*?*

Looking at $p(x)=x^2(x-5)+5x(x-5)+23(x-5)+116$ it seems awfully tempting to put in $x=5$. Doing so gives

$$p(5)=0+0+0+116.$$

We see that the number 116 is the value of the polynomial at $x=5$.

We also see from this example that dividing $p(x)$ by $x-5$ gives a remainder of 116, which is $p(5)$.

Challenge: Compute $\dfrac{x^n-1}{x-1}$ via this method. Explore $\dfrac{x^n-1}{x+1}$.

Playing with examples makes the following observations very clear.

Given a polynomial p *and a linear term* $x-h$, *we can always write* $p(x)$ *as multiples of* $x-h$ *plus a single number* r.

$$p(x)=(\text{multiples of } x-h)+r$$

Putting in $x=h$ *shows that* $r=p(h)$.

This means that dividing $p(x)$ *by* $x-h$ *gives a remainder equal to* $p(h)$.

Pushing the logic a bit further we see $p(x)$ *is evenly divisible by* $x-h$ *if, and only if,* $p(h)=0$.

This leads to the Zero Theorem of polynomials: *If* $p(h)=0$ *for a polynomial* p, *then* $p(x)$ *has* $x-h$ *as a factor.*

Challenge: Can you get creative and use this wishful thinking approach to find a formula for $\dfrac{1}{1-x}$? (How can one adjust "1" so that it looks like a multiple of $1-x$, and then adjust any new terms that might appear?)

All is indeed ripe to help students learn how to play like mathematicians!