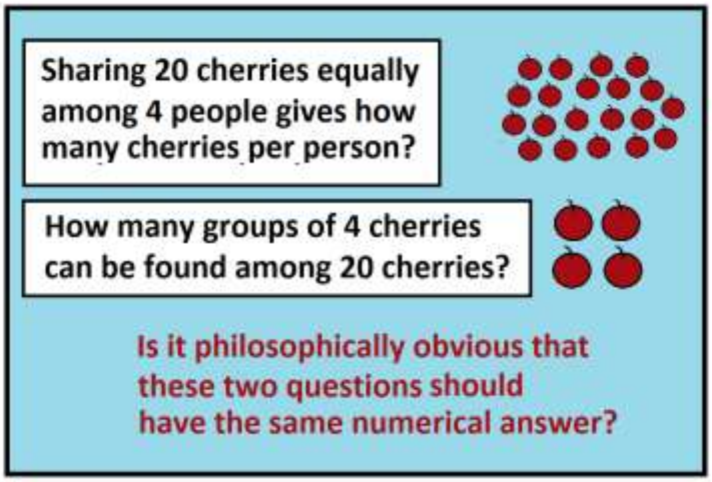


TEACHING THE PROBLEM-SOLVING MINDSET

A Classroom Moment

MATHEMATICAL ASSOCIATION OF AMERICA

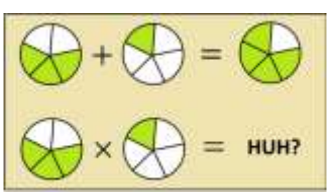
BY JAMES TANTON

WWW.MAA.ORG/CI

Dealing with Fractions

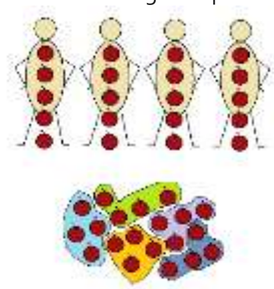
Fractions are hard! The story of fractions, as presented to our students, is subtle and very confusing.

Fractions are typically first defined as "parts of a whole," such as proportions of pie. And in this model it makes perfectly good sense to add fractions: three-fifths of a pie plus one more fifth gives a good serving of pie. It makes absolutely no sense, however, to multiply fractions in this context.



At some point students might be led to the mantra: "of means multiply." And we can draw a picture of one third of half a pie. But does this properly justify writing $\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$? (Also, is it philosophically obvious that half of a third of a pie should yield the same amount of pie?) Why should "of" mean multiply?

Fractions are, eventually, seen as numbers in the own right: they are answers to division problems. But what is division? Division can be interpreted at least two ways: as the equal sharing of goods and as the process of identifying subgroups of a given size. Is it obvious that the two questions in the letterhead have the same numerical answer? There is subtle reasoning to explore!



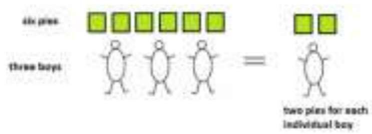
[To answer this… If 20 cherries are shared equally among four people, then each first cherry assigned to a person makes a group of four cherries, each second cherry another group of four, and so on. Thus there are as many groups of four as there are cherries assigned to each person. Conversely, for a given group of four cherries, we can assign one cherry from the group to each person. Thus each person receives as many cherries as there are groups of four.]

I believe that a deliberate, unhurried review of fractions should be a standard part of a high-school curriculum. There is so much hazy thinking associated with fractions and high-school students finally have the perspective and the intellectual readiness to confront that haziness and work their way through it. This is the time to finally learn fractions! Moreover, a study of fractions at this point brings to light the core component of master learning and problem solving: to actively work to identify hazy thinking, to deliberately confront the haze and not shy away from it, and to do one's concerted utmost to clear that fog away!

(For a full outline of the fraction story and its complex and hidden subtleties, as well as one approach to helping high-school students find context and coherence within the story, see my online notes www.gdaymath.com/courses/fractions-are-hard/ .)

The Key Fraction Property

The different models of fractions we present to young students serve to motivate the different fraction rules we feel compelled to believe. Consider, for instance, the "pies per boy" model. Here a fraction $\frac{a}{b}$ is interpreted as the amount of pie an individual boy receives when a pies are shared equally among b boys. (So, for example, $\frac{6}{3}$ represents the result of sharing six pies among three boys, and so is 2, two pies per boy; and $\frac{1}{2}$ is the result of sharing one pie among two boys. We call this result "half" a pie.)



The fraction $\frac{6}{3}$ equals 2.

Here we see in a given pie sharing situation that if we double the number pies and double the number of boys, the amount of pie each individual boy receives is unchanged:

$\frac{2a}{2b}$ has the same value as $\frac{a}{b}$.

Similarly, the amount of pie per boy is unchanged if we triple the number of pies and triple the number of boys, or halve the number of pies and of boys, and so on. Thus this model suggests a property of fractions, that

$$\frac{xa}{xb} = \frac{a}{b}$$

for all (real?) numbers x. This is a key property. A high-school course on fractions should derive it properly rather than appeal to a specially selected model of fractions.

PROBLEM-SOLVING MOMENTS

No one really likes to work with fractions, even after finally learning in a high-school course what they actually are. So the best way to deal with them might still be to avoid them! Fraction work thus provides an opportunity to engage in the general problem-solving strategy: **AVOID HARD WORK!**

Some examples.

Example: Make $\dfrac{4\frac{1}{3}}{2\frac{2}{5}}$ look friendlier.

Answer: Fractions within fractions are particularly irritating. Can we avoid them? Yes! Let's first focus on that third in the numerator. We can eliminate it by multiplying the numerator and denominator each by three. (By the key fraction property this does not change the value of the fraction.)

$$\frac{\left(4+\frac{1}{3}\right)\times 3}{\left(2+\frac{2}{5}\right)\times 3} = \frac{12+1}{6+\frac{6}{5}}.$$

Let's contend now with that fifth. Multiply the numerator and denominator each by five.

$$\frac{(12+1)\times 5}{\left(6+\frac{6}{5}\right)\times 5} = \frac{60+5}{30+6} = \frac{65}{36}.$$

Is this friendlier? We have shown that sharing $4\frac{1}{3}$ pies equally among $2\frac{2}{5}$ girls gives the same amount of pie per girl as does sharing 65 pies among 36 girls. That's just over one and three-quarter pies per girl.

Comment: Of course, we could have been swift and multiplied top and bottom each initially by 3×5. Let students suggest this.

Example: Compute $\frac{7}{3} \div \frac{5}{8}$.

Answer: We are being asked to make $\frac{7/3}{5/8}$ look friendlier. The natural thing to do is to multiply numerator and denominator each by three,

$$\frac{\left(\frac{7}{3}\right)\times 3}{\left(\frac{5}{8}\right)\times 3} = \frac{7}{\left(\frac{5\times 3}{8}\right)}$$

and then each by eight,

$$\frac{7\times 8}{\left(\frac{5\times 3}{8}\right)\times 8} = \frac{7\times 8}{5\times 3}.$$

The answer $56/15$ appears.

Comment: I personally teach the division of mixed numbers first. Then there is no special rule to be memorized for dividing fractions: they are just a special case of mixed number division.

For students who wonder about the "multiply by the reciprocal" rule, we can ask if they see it in play in the given solution. (There are advantages to delaying all arithmetic computations.) I then advise them to forget this rule.

Example: Compute $4.5 \div 0.009$.

Answer: Dealing with decimals within fractions is hard. Let's avoid the hard work!

$$\frac{4.5}{0.009} = \frac{4.5\times 1000}{0.009\times 1000} = \frac{4500}{9} = 500.$$

Example: Solve for x in $\frac{x}{5} = \frac{7}{3}$.

Answer: In a "rote mindset" students might cross-multiply to write $3x = 35$, and then divide each side by three to get $x = \frac{35}{3}$.

This is, of course, correct, but why multiply each side by three if one is only going to next divide through by three?

The equation $\frac{x}{5} = \frac{7}{3}$ has x locked as a numerator of a fraction. Can we avoid this? Well, multiplying through by its matching denominator does the trick. We get

$$\frac{x}{5}\times 5 = \frac{7}{3}\times 5,$$

that is, $x = 35/3$.

Comment: Let's eradicate cross-multiplication from the curriculum! Instead, let's encourage students to engage in the problem-solving mindset and conduct carefully selected, helpful actions of their choice.

As educators, we are all well trained to "clear fractions" in complicated equations and do so almost without thinking. For instance, in solving for x in

$$\frac{x}{3} + \frac{5}{12} = \frac{7}{2} - \frac{5x}{6}$$

we would each instinctively multiply through by twelve. (Does "cross-multiplying" make sense here?) One can, of course, work through the fraction arithmetic, and many beginning students might do this. It is unlikely to be joyful.

We can take opportunities like these to encourage students to pause and think before leaping into action. As a mathematician I often "work hard to avoid hard work." Simply asking if there a way to make an equation or expression look friendlier is usually enough to encourage students to do the same.