TANTON'S TAKE ON …

Significant Figures

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If a scientist told you that she measured the length of a needle to be 3.47 cm, then you know that she measured that length to the nearest one-hundredth of a centimeter.

If, later, a second scientist came to you and said she measured the length of what happened to be the same needle and came up with the answer 3.5 cm, you would know that she measured only to the nearest tenth of a centimeter (and you would say that her result is in agreement with the first measurement).

The digits "3" and "4" and "7" were each significant to the first scientist's measurement (she was working to the accuracy of hundredths), and the digits "3" and "5" were significant to the second's (working just in tenths).

Suppose a third scientist now came along and gave the measurement of the needle as 3.470 cm. Mathematically the numbers 3.47 and 3.470 are identical, but by writing down that final zero this scientist is saying "I measured to the nearest thousandth and got the answer 3 cm, 4 tenths, 7 hundredths, and 0 thousands." She has four significant figures: 3, 4, 7, and 0.

When conducting a measurement, the count of digits that correspond the degree of accuracy of your measurement are called the *significant figures* of the measurement.

Saying that sounds good. But as a definition— of sorts—this is a bit vague and is certainly confusing if the smallest units of measurement are very large or very small.

For example, suppose a town's population is recorded as 48,000 people. Is this measured to the nearest thousand (in which case there are only two significant figures: 4 and 8)? Or to the nearest hundred (three significant figures: 4 and 8 and the first 0)? Or to the nearest ten, or to the nearest person?

Comment: Some people like to use the decimal point at an end of an integer quantity to indicate that the count was to the nearest unit. For example, writing the population count as "48000." means that the count was the level of the individual: there are, right on the nose, forty-eight thousand people in that town!

At the other end of the spectrum, a measurement of 0.00047 grams, for example, suggests that there are zero tenths of a gram, zero hundredths, zero thousandths, four ten-thousands and seven one-hundred-thousandths of a gram measured, with all five digits being significant. (Or is it six digits? There are zero whole grams recorded too!)

CONVENTIONS

As we can see, it is really the zeros listed in measurements that are confusing. Look at any textbook or website on the matter of significant figures and you will see a list of conventions on how to handle the zero digits. The rules are of the ilk...

1. *All non-zero digits are considered significant.*

This should be stated.

2. *Zeros appearing between non-zero digits are considered significant.*

For example, 120029 and 450.003 are each said to have six significant figures.

3. *Leading zeros are <u>not</u> significant.*

For example, 0.00012 is said to have only two significant figures. (More on this in a moment!)

4. *Trailing zeros after a decimal point <u>are</u> significant.*

For example, 0.1200 has four significant figures.

5. *To cope with trailing zeros before a decimal point, use a bar to denote the last significant figure.*

For example, $48\bar{0}00$ indicates that the first zero is significant. This measurement was completed to the nearest hundred and there are three significant figures in all.

ABOUT CONVENTION THREE:

This convention is a source of confusion and contention!

For example, suppose Sally said she measured the width of a needle to be:

$$0.000012 \text{ km}$$

She argues it really was zero kilometers, and zero tenths of a kilometer, and zero hundredths of a kilometer, and so on, and so her zeros truly are significant. (After all, you would say that a measurement of 0.030012 km would have a significant three in it.)

But Ralph says that she was silly to measure the width of a needle in terms of kilometers, and should have used millimeters instead. In which case she should express her answer as:

$$1.2\,\text{mm}$$

Here only the 1 and the 2 are significant.

Rule three is saying that for really small numbers we are going to assume that the measurer used the appropriate size of unit for the measurement and that we'll consider only the figures that are significant for that sized unit. Whether or not this is reasonable, or even appropriate, is completely debatable!

THE WAY TO AVOID CONFUSION AND TO AVOID MEMORIZING CONFUSING CONVENTIONS

Just do what scientists actually do!

Working with very large and very small numbers can be unwieldy and scientists prefer to express values in <u>scientific notation</u> in order make their comprehension and manipulation manageable.

Each number is expressed as a *single digit* followed by a decimal point and some decimal digits, all multiplied by a power of ten:

$$a.bcd\ldots\times 10^{e}$$

For example, to practice the interplay:

$$3.5\times 10^{4} = 3.5\times 10\times 10\times 10\times 10 = 35000$$

$$5.43\times 10^{5} = 5.43\times 10\times 10\times 10\times 10\times 10 = 543000$$

$$9.1\times 10^{-2} = 9.1\times\frac{1}{10}\times\frac{1}{10} = 0.91\times\frac{1}{10} = 0.091$$

and

$$0.045 = 0.45\times\frac{1}{10} = 4.5\times\frac{1}{10}\times\frac{1}{10} = 4.5\times 10^{-2}$$

$$678000 = 6.78\times 100000 = 6.78\times 10^{5}$$

$$0.000009 = 9\times\frac{1}{10}\times\frac{1}{10}\times\frac{1}{10}\times\frac{1}{10}\times\frac{1}{10}\times\frac{1}{10}$$
$$= 9\times 10^{-6}$$

Comment: Many people are taught to count places to the left and right of the decimal point and memorize a rule as to which power of ten these correspond. This is difficult. It is conceptually easier to multiply and divide by single tens and just count them. (You, and your students, will naturally get speedier at this the more you do it – if speed is important to you.)

If one uses scientific notation, then the five conventions for significant figures boil down to one simple idea:

Use scientific notation! Every digit mentioned to the left of the power of ten is considered significant. (And don't ever mention the word "significant.")

For example,

Write $48\bar{0}00$ as 4.80×10^4. It is then clear that you meant the digits 4 and 8 and just the first 0 at the hundreds level.

Write 0.00012 as 1.2×10^{-4}. (And this explains why wise powers deem leading zeros after a decimal point as insignificant.)

Write 0.1200 as 1.200×10^{-1} and its four significant figures are clear.

DOING ARITHMETIC WITH NUMBERS FROM SCIENCE CLASS

High-school education folk, who teach the language of significant figures, have developed an (almost) standard practice for manipulating quantities representing measurements. The practice represents a ROUGH AND READY approach that gives a quick sense of how errors might propagate as you add, subtract, multiply and divide quantities. (It is also designed to help students break the temptation of thinking that just because they can perform a calculation to a large number of decimal places it means that the level of precision has increased! For example, 3.14×2.76 has an answer with four decimal places, even though the original numbers are measured only to hundredths.)

The convention here varies from author to author, from website to website. (So watch out!) But basically all versions of the practice boil down to the following instructions:

When adding, subtracting, multiplying, or dividing quantities:

1. First do the arithmetic of the calculation without regard significant figures.

2. Round the answer to the position of the rightmost place of the term with the least number of significant digits. (For example, if one term is measured to tenths, say, and the rest are to hundredths, then round the final answer back to tenths.)

(Most authors actually give more sophisticated rules for handling products and quotients. They'll cringe at my simplicity here.)

Some examples:

$113.6 + 21.09 = 134.69 \to 134.7$
(Rounded to tenths)

$3.1 + 2 = 5.1 \to 5$
(Rounded to units)

$1\bar{1}00 + 1\bar{2}80 = 1380 \to 1\bar{4}00$
(Rounded to hundreds)

$200. + 1201. = 1401 \to 1401.$
(Rounded to units)

Some more examples:

$7 \times 7 = 49 \to \bar{5}0$
(The final answer is to one sig fig)

$\frac{6}{3.0} = 2 \to 2$
(The final answer is to one sig fig)

$\frac{6.0}{3.0} = 2 \to 2.0$
(The final answer is to two sig figs)

$2 \times 0.9 = 1.8 \to 2$
(The final answer is to one sig fig)

Still more examples:

$$\frac{5.20}{1.304} = 3.987730061... \to 3.99$$

$$\left(3.2 \times 10^{13}\right) \times \left(2.01 \times 10^{20}\right) = 6.432 \times 10^{33} \to 6.4 \times 10^{33}$$

$$\left(3.2 \times 10^{7}\right) \times \left(1.6 \times 10^{5}\right) = 5.12 \times 10^{12} \to 5.1 \times 10^{12}$$

WHAT SICENTISTS REALLY DO:

Scientists are clear and explicit about the errors in their measurements and they are clear and explicit about how those errors affect calculations.

They handle their calculations in a way that is best and most appropriate for the problem at hand, and communicate—well—exactly how they chose to handle the errors.

There is no standard convention among scientists as to what procedure to follow (other than the convention to use scientific notation to express numbers). The only rule is: **Be intelligent and do what makes sense for the context at hand.**
And I say: Let's teach this! (I MEAN IT!)

For example, suppose a scientist uses a ruler marked with millimeters to measure the sides of a rectangle. She computes:

$$\text{width} = 12.5 \text{ cm}$$
$$\text{length} = 130.3 \text{ cm}$$

She will describe the ruler she used in her published article and actually write in her article something like.

$$\text{width} = 12.5 \pm 0.05 \text{ cm}$$
$$\text{length} = 130.3 \pm 0.05 \text{ cm}$$

because she could only measure to the nearest millimeter.

To compute the area of the rectangle, she would say:

The largest the area could be is:
$$12.55 \times 130.35 = 1635.8925 \ cm^2$$
The least the area could be is:
$$12.45 \times 130.25 = 1621.6125 \ cm^2$$

That is, her measurements and calculations show that the true area of the rectangle is somewhere between these two values.

High praise to a young student who would express her work in this sort of way too!

Comment: Just for comparison, a student trained in "significant figure" thinking would compute the area of the rectangle as:

$$12.5 \times 130.3 = 1628.75 \rightarrow 1628.8 \ cm^2$$

This is indeed a "rough and ready" version of the same work.

A SMALL PEDAGOGICAL COMMENT:

The use of scientific notation is too cumbersome to introduce in typical high-school laboratory work: measurements made in a lab class are usually "human scale" and are expressed in just units or tens of units, and with two or three decimal places at most (and not with giga units or nano units). Writing 12.3, for example, makes more sense than writing 1.23×10^1. However, the analysis of errors I discuss here, avoiding the fuss with significant figures, is worth considering. So "12.3" should really be written 12.3 ± 0.05. (Or, to save ink, a comment like the following could be made at the top of a lab-book page: *All data values in this table have error range* ± 0.05 *cm.*)

A BIG STEM COMMENT:

Have you ever pondered what it must be like for a student going back and forth between their science and mathematics classes?

In one class a student might be penalized for failing to compute 3.14×2.76 to all four decimal places, and in another, penalized for doing so!

In one class a student would never dream of combing x and x^2 via addition to form $x + x^2$ ("*You can't add a length and an area*!") And yet in another class they are not meant to blink an eyelid at this practice, and, in fact spend weeks dwelling on these sums without a question to be had. (These weeks are called "a unit on quadratics" in algebra class.)

And a deep thinking student (like all students are, if given the psychological permission to think threatening questions) might wonder …

If $2\,cm \times 3\,cm$ *is the area are of rectangle* ($6\,cm^2$) *and* $2 \times 3\,cm$ *is two copies of a length of three cm (giving* $6\,cm$*), what is plain old* 2×3*? (Actually, what is multiplication? Does the "*$\times$*" symbol mean the same thing in all three cases?)*

Many schools are working very hard to "be STEM," developing innovative, integrated programs. I hope that equal attention is being given to developing integrated, philosophical clarity at this base level too. This is important and key to true STEM success. To me, STEM is about developing the art of enquiry and the confidence to follow through on enquiry. This means questioning basic assumptions and one's own fundamental understanding. By <u>not</u> connecting philosophical mismatches between science and mathematics departments, we teach students instead the art of sweeping thorny ideas aside and the practice of carrying on through matters with only hazy thinking at hand. That is antithetical to STEM.

Does multiplication in science class mean the same thing as multiplication in mathematics class?

Are there three different formulas relating distance, time and speed in physics class, but only one in math class?

If, in teaching a method of for converting units, the term "$\frac{1\,foot}{12\,inches}$" is referred to as a "ratio of one," should a math student be jarred?

What is a student meant to think if she learns that $\sin\left(90^\circ\right)$ has no meaning in her geometry class (a right triangle cannot possess a second right angle), yet she is expected to compute the sine of obtuse angles nonetheless in her physics class?

What if she models the motion of a falling ball via $y = -16t^2 + 20t + 10$ in science, but later plots functions like these as U-shaped graphs in math class? (There is no "U" for an object falling straight down!)

Is time the fourth dimension? Science classes seem to lean towards a "yes" answer. Math classes don't seem so committed.

Can we keep an eye out for points of confusion and philosophical mismatch between our departments? Can we keep in mind the take-away messages students are actually being left with after a day of walking between our classes? Can STEM work to integrate the absolute fundamentals too and teach its key message at the subtle levels?

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