# TANTON'S TAKE ON …

# "0.9999…"

## CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

**SEPTEMBER 2012**

> It is a classic conundrum that comes up in most every math classroom:
>
> *The quantity* $0.999\ldots.$
> *Does it or does it not equal one?*

**Albert:** It can't equal one.

Think about it. 0.9 is smaller than 1, and 0.99 is smaller than 1, and 0.999 is smaller than 1. And so is 0.9999999999 999999999999999999999999999999 999999999999999999999999999999 999999999999999999999999999999 99999999999999999999. Even if we write infinitely many nines we'd still be smaller than one, by some infinitesimal amount. We never get to one!

**Bilbert:** I am not sure. We all know that one third has the decimal expansion with recurring threes:

$$\frac{1}{3} = 0.33333\ldots.$$

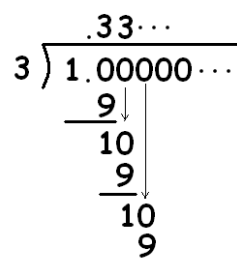
Triple each side:

$$3 \times \frac{1}{3} = 3 \times 0.33333\ldots$$

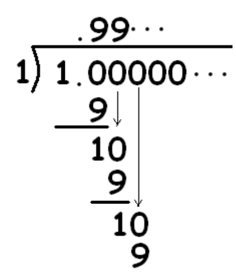
and we see $1 = 0.99999\ldots.$ We got to 1.

**Albert:** Why do people say that $0.3333\ldots$ equals one third? The argument I offered before also applies here! Clearly 0.3 is not one third – it is too small. Nor is 0.33 one third – also too small. So too are 0.3333 and 0.33333333; they are each is just under one third. Again, even if we write down infinitely many threes we'll be infinitesimally shy of one third.

**Bilbert:** But $0.3333\ldots$ <u>is</u> one third! I can see it is so if I do the long division.




In fact ... I can even see that $0.999\ldots$ has to be 1 if I compute $1 \div 1$ in a clever way. Instead of saying that "1 goes into 10 times" let's say that "1 goes into 10 just nine times" and carry the 1.




Voila! $1 \div 1 = 0.9999\ldots.$

**Albert:** "Carrying the one." Come on! One goes into 10 ten times!

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**Cuthbert**: Bilbert is right. What he is really doing here is an algebraic argument of some kind. Maybe something like the following helps?

Let's give $0.999...$ a name, say, $D$ for Dilbert.

$$0.999... = D$$

Multiply each side by ten to get:

$$9.9999.... = 10D$$

The left side is $9 + 0.9999...$ which is $9 + D$. So our equation reads:

$$9 + D = 10D$$

Algebra now gives $D = 1$.

**Albert:** Hang on! Be it algebra in disguise or algebra directly, these arguments just cannot be right! Let me play you at your own game.

Consider $...999999$, a number with infinitely many nines to the left. (It's really $9 + 90 + 900 + 9000 + \cdots$.) Give it a name, $E$ for Egbert, say.

$$....99999 = E$$

Multiply each side by ten:

$$...999990 = 10E$$

The left side is what we had before, minus nine, so this reads:

$$E - 9 = 10E$$

Thus $-9 = 9E$ and $E = -1$. Are you telling me that $9 + 90 + 900 + 9000 + \cdots$ adds up to <u>negative one</u>??!!

**Bilbert:** Hmm. Albert has a point. Something is awfully fishy.

**Cuthbert:** Hmm. I agree that there is something to think about here. What if we had the number with infinitely many nines to the left AND infinitely many nines to the right: $...9999.9999....$? Let's call it $F$ for Filbert and apply the same algebra:

$$...9999.9999... = F$$

Multiplying by ten moves the decimal point, but that gives back the same number!

$$...99999.999... = 10F$$

And so $F = 10F$ giving $F = 0$.

**Bilbert:** And this is consistent! If $0.9999... = 1$ and $...9999 = -1$ then

$$\begin{aligned} ...9999.9999.... &= ...9999 + 0.9999... \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

I don't know what any of this really means but it is hanging together.

**Albert:** But surely you don't believe that $...9999$ equals $-1$?

**Bilbert:** You are right, I don't. But I do want to believe that $0.999....$ equals $1$, even though the algebra now seems suspect to me.

**Cuthbert:** What about this?

$$\begin{array}{r} {\scriptstyle 1\,1\,1}\phantom{99} \\ ...999999 \\ +1 \\ \hline ..000 \end{array}$$

Is this sum saying that "$...9999 = -1$" is right?

**Bilbert:** My brain hurts!

**Cuthbert:** Maybe we should come at this another way? I remember being taught in school the *geometric series formula*:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

**Albert:** It is an infinite sum. I don't trust math with infinity!

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**Cuthbert:** I know. I've always approached math with skepticism. I had a teacher who really encouraged me to pause and wonder for myself if formulas and answers to questions seemed valid. I didn't personally believe the geometric series formula until I came up with a paper-sharing idea that convinced me. Let me show it to you.

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*Cuthbert holds up a piece of paper.*

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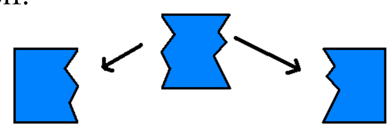
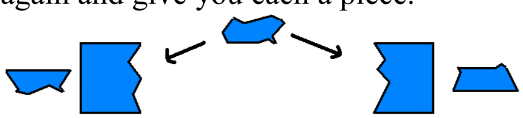
Here's a piece of paper which I am tearing into thirds. I'll give each of you a third and keep the remaining third for myself.

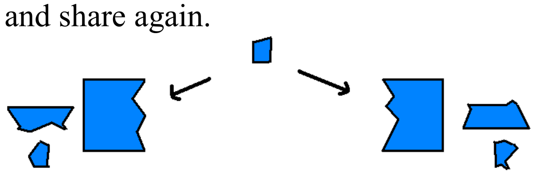


Actually, I am a generous guy, and so I am going to tear my third into thirds again and give you each a piece.



Right now you each have a third, plus a third of a third ( $1/3+1/9$ ) and I am holding onto a small piece one-ninth in size.

Actually let me tear my ninth into thirds and share again.



Now you each have $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$ of paper and I have a tiny amount – which I'll share again by tearing into thirds.

Now imagine I do this forever. The amount of paper I possess dwindles away to nothing and the paper is being equally shared among you both.

Now ask: In the end, how much paper will you each get?

The math says that you will each receive $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$ paper. But logic says that since <u>all</u> the paper is shared equally between you both, you each get half the paper. Thus:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{2}.$$

This is the geometric series formula with $x=1/3$ (and the beginning one subtracted from the right).

In the same way if I share a piece of paper with $19$ friends by tearing it into 20 pieces and divvying up my share over and over again we'll see:

$$\frac{1}{20}+\left(\frac{1}{20}\right)^2+\left(\frac{1}{20}\right)^3+\cdots=\frac{1}{19}$$

And in general:

$$\frac{1}{N}+\left(\frac{1}{N}\right)^2+\left(\frac{1}{N}\right)^3+\cdots=\frac{1}{N-1}$$

**Albert:** Umm …So … You only believe the geometric series formula only for certain values of $x$, namely those that are fractions of the form $\frac{1}{N}$?

**Cuthbert:** Fair enough. You are right. Maybe I should still be skeptical of the formula for other types of values for $x$.

**Bilbert:** Oooh … What if you share a piece of paper among yourself and two-and-a-half friends? Divide the paper into seven parts: give two people two parts, a third person one part, and keep two parts for yourself. Then repeat this sharing process. I bet you can see the formula holds for other $x$-values as well.

**Cuthbert:** Great idea! Sounds like a super project.

But for now, let's put $N=10$ to see

$$\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots=\frac{1}{9}$$

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This is just paper-sharing with nine friends and so is a formula I believe. Multiply through by nine:

$$\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots=1$$

This is saying: $0.999...=1$.

**Albert:** I like paper-tearing, and I like Bilbert's idea of extending it to all sorts of numbers to get to the general geometric series formula. But the formula itself still cannot be right! For example, put $x=10$ into it to get:

$$1+10+10^2+10^3+\cdots=\frac{1}{1-10}$$

That is, $1+10+100+1000+\cdots=-\frac{1}{9}$.

Nonsense! So why should I believe the formula when it says

$\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots=1$?

**Bilbert:** But look! Multiply your "incorrect" formula

$$1+10+100+1000+\cdots=-\frac{1}{9}$$

though by $9$ and get:

$$9+90+900+9000+\cdots=-1.$$

We're back to $...9999=-1$! Everything is consistent – if not strange and brain-hurty!

**Cuthbert:** That's basically my reason for being skeptical of the geometric series formula. When I first saw it I put in $x=1$ and wondered about

$1+1^2+1^3+\cdots$ equaling $\frac{1}{0}$ .

**Bilbert:** My algebra from before says that the geometric series formula is right!

Give $1+x+x^2+\cdots$ a name, say, $H$ for Hilbert, and multiply through by $x$:

$$1+x+x^2+\cdots=H$$
$$x+x^2+x^3+\cdots=xH$$
$$1+x+x^2+x^3+\cdots=1+xH$$
$$H=1+xH$$
$$H=\frac{1}{1-x}$$

Thus $1+x+x^2+x^3+\cdots=\frac{1}{1-x}$.

I am willing to say that one divided by zero is infinity and so $1+1^2+1^3+\cdots$ equaling $\frac{1}{0}$ is good! And I guess I am just going to have to believe that $..9999$ is $-1$. It's strange, but the math is consistent.

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*Silence follows.*
*Eventually Albert chimes in.*

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**Albert:** I think Bilbert hit on the crux of the matter when he says "believe."

The problem is that we are playing a mind game, doing something that is beyond human.

Think of it. Even your paper-sharing argument, which we all seem to like, is beyond human: we will never actually see you dispel <u>all</u> of the paper from your hand, not in our lifetimes, and not in all of geologic time. In theory, there will always be some paper, smaller than the smallest of subatomic particles, in your possession and the process of sharing will continue forever on. We will never ever actually see <u>all</u> the paper distributed between us.

So I guess I am back to my original argument. We as humans never see $0.999...$, we only ever see finite versions of it, $0.9$ or $0.99999$ or $0.9999999999$. We've been grappling with something that is actually beyond human.

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Yes. That is the problem. Have you noticed that we haven't even written down what we are talking about?! We cheat and write "..." when we really should be writing down an infinite number of nines. And we cheat because we can't do it. We are human! All this is really a mind game. We're asking "What if we could go beyond the end of time and truly see an infinite number of nines, or truly see the completion of Cuthbert's paper sharing?" We seem to be going back and forth about whether or not we are willing to play this "beyond human" mind game.

**Bilbert:** But what about the math? The algebra gives us definite answers!

**Albert:** Well think about it. We started each time by giving the quantity a name, which is really an assumption that there is an answer that can be named to begin with! The math isn't lying to us, it is just saying:

***IF*** *you believe that* $0.9999...$ *has a meaningful answer, then that answer has to be* $1$.

***IF*** *you believe that* $...9999$ *has a meaningful answer, then that answer has to be* $-1$.

***IF*** *you believe* $1+x+x^2+x^3+\cdots$ *has a value, then that value must be* $1/(1-x)$.

The math itself is making no claim as to whether or not these quantities should have meaningful answers to begin with. That's up to us!

**Bilbert:** Aha! So I am right … Since I am willing to believe that $0.9999...$ is a meaningful quantity, it does indeed equal 1, for sure, right on the nose!

**Cuthbert:** And Albert is right too. He is arguing that there is doubt to be had on whether or not an infinite decimal expansion, of any kind really, has meaning. We, as humans, only ever see finite versions of infinite quantities. He's saying "What is the value of $0.999...$?" could simply be a meaningless question.

**Albert:** Exactly. At first I was arguing that $0.999...$ doesn't equal $1$. But I was operating under the same assumption as Bilbert, that there was a meaningful answer to argue about in the first place. It is not "Does $0.999...$ equal one or does it not?" but rather "Does $0.999...$ have a meaningful value or does it not?"

**Bilbert:** I agree. Although I said earlier that I am willing to accept that $...9999$ is $-1$, I didn't really believe it. I think that $...9999$ really can't have a meaningful value. I am on Albert's side for this one.

**Cuthbert:** I learnt in computer science, where all numbers are expressed in base two, that 1111111111111111 is assigned the value $-1$. This is their version of $...999$. Clearly computer scientists have decided this number has meaning and should have value $-1$! It's all context!

I bet mathematicians have created all sorts of arithmetic systems in which quantities like $...9999$ are defined. And we've proved here that in every one those systems $...9999$ equals $-1$ every time! (As long as the same basic laws of algebra still hold.) Context! Context!

**Bilbert:** So in the system of real numbers and arithmetic as we know it, how do we decide when we should believe "beyond human" answers?

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*In walks the college professor.*

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**Professor:** Welcome to Calculus.

**Short Videos to Watch:**

*Developing the Geometric Series Formula through Paper-Tearing:*
www.jamestanton.com/?p=723

*Does 0.999… equal 1 or does it not?*
www.jamestanton.com/?p=350

*A Few Thoughts on Troublesome Zero*
www.jamestanton.com/?p=1084
(I add this because at one point Bilbert did say that he was willing to believe that "one divided by zero is infinity.")

**Books:**
The ideas discussed here also appear in *CHAPTER 1: Leaps of Faith* in *THINKING MATHEMATICS! Volume 6: Calculus* available at www.lulu.com.

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