



TANTON'S TAKE ON ...



THALES'S THEOREM

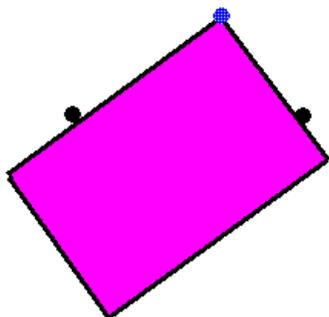


OCTOBER 2013

Let's start with an action puzzle:

PAPER PUSHING:

Draw two dots on a white board and slide a piece of paper up between the dots as shown. Mark the position of the corner of the paper.



Now slide the paper up between the dots at a different angle and mark where the top corner lies this time. Repeat at least 15 more times.



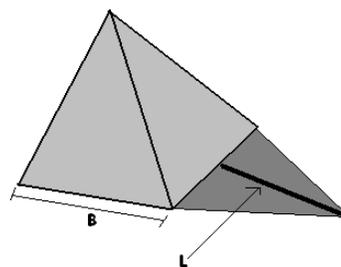
What shape curve does the trace of dots seem to describe? Can you prove any claims you make?

Thales of Ionia:

Geometry, like all of mathematics, is an intensely human enterprise. Its name, from the Greek *ge* for "earth" and *metria* for "measure," suggests that geometry evolved from practical concerns – the art

of accurately measuring tracts of land, performing feats of navigation, and accomplishing success in construction and architectural design, and the like. But as soon as one starts a study of the subject, one is drawn from the products of the hand to beautiful and profound products of the mind. Geometry is a gateway to a universe of intellectual richness and surprise.

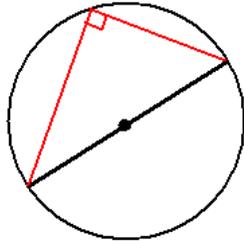
Greek mathematician Thales of Ionia (ca 600 B.C.E.) is dubbed the "Father of Geometry" for his first attempts to identify and establish geometric theorems (some 300 years before Euclid). The fellow impressed the Egyptians by calculating the height of the Great Pyramid from measuring the length of its shadow. He did this on the day of the year that the Sun set directly behind the pyramid (so that the shape of the shadow was an isosceles triangle) at the exact time of day that the length of his own shadow matched his own height.



$$\text{Height} = L + \frac{B}{2}$$

Thales is also credited as the first to explicitly detail a logical proof of a geometric result. Thales' theorem, as it is known today, states:

Whenever an angle is drawn from the diameter of a circle to a point on its circumference, then the angle formed is sure to be a perfect right angle.



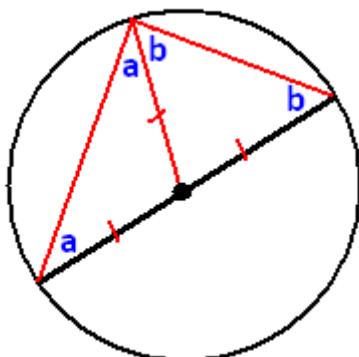
Proving Thales' Theorem:

Thales was at the point of his understanding in geometry to believe two things:

- The three interior angles of any given triangle sum to 180° .
- The two base angles of an isosceles triangle are congruent.

If you are at a point in your understanding to believe (or be able to prove?) these two points too, then you are all set to establish Thales' theorem.

Draw in an extra radius and mark congruent base angles of the isosceles triangles formed.



We see: $a + (a + b) + b = 180^\circ$.

Thus $2a + 2b = 180^\circ$ giving $a + b = 90^\circ$.
We have a right angle.

Back to Paper Pushing:

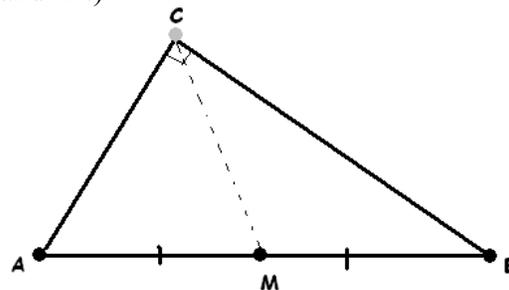
If you conduct the paper-pushing experiment it is very tempting to believe that the curve emerging before your very eyes is a perfect semi-circle. Fortunately our intuition is correct! (That is not always the case in geometry.)

This claim does not follow from Thales' theorem. Thales starts with a circle and shows the construction of a right angle from it. The paper-pushing experiment starts with a right angle and constructs, allegedly, a circle from it. It is, in some sense, the converse of Thales' theorem and requires its own proof.

Creating (and notice I said "creating") a proof of the paper-pushing claim makes for a great student discussion.

If we think the curve is a perfect semi-circle, where do you think the center of that semicircle lies?

Most everyone suspects the center to be the midpoint of line segment connecting the two original dots on the board. If we label those points A and B , and the midpoint M , we are led pondering the properties of a diagram like the following. (Here the point C is the corner of the paper pushed between A and B .)



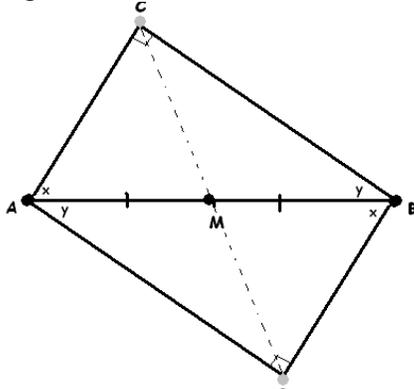
What would we like to be true in this diagram? What would establish that a plot of points like C is a semi-circle?

We need to show that the point C is the same distance from M as each of A and B are.

So, can we prove MC equals MA and MB ?

At this point I stop talking and wait for someone in the class to have a brilliant flash of insight. (And I assure you, that flash of insight has never failed to come – usually after about 20 minutes of silence or mostly silence!)

Here's the flash: Draw a rotated copy of triangle ABC like so.



This figure looks like a rectangle with segment \overline{MC} half of one of its diagonals.

To prove it is a rectangle, notice that the two angles x and y in the original triangle sum to 90° . Consequently all four interior angles of the quadrilateral we see are indeed right angles.

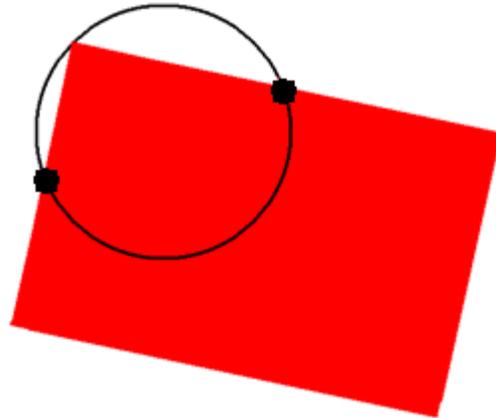
Since the figure is a rectangle we know its diagonals bisect each other and are equal in length. Consequently, the length MC does have the same value as lengths MA and MB .

(The Pythagorean Theorem shows the diagonals of a rectangle are the same length and congruent triangles – look for congruent alternate interior angles – establish that the diagonals bisect one another.)

The properties of quadrilaterals establish a beautiful result about the construction of circles. How delightful!

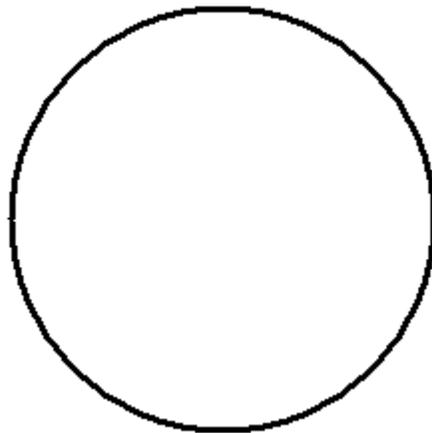
Here's how you find the diameter of a drinking glass:

Trace the rim of the drinking glass on a piece of paper. Lay the corner of the second piece of paper on the circle you just drew as shown.



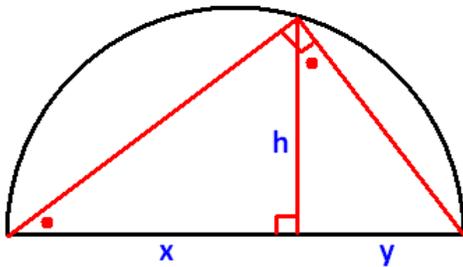
Mark where the edges of this paper intersect the circle and measure the distance between those two points. This is the diameter of your drinking glass.

Exercise: Using nothing more than a piece of paper and a pencil mark, with high accuracy, the center of THIS VERY CIRCLE!



An extra result:

Consider a right triangle constructed on the diameter of a circle as shown:



Suppose the altitude of this right triangle divides the diameter into two parts of lengths x and y . Show that the altitude has length:

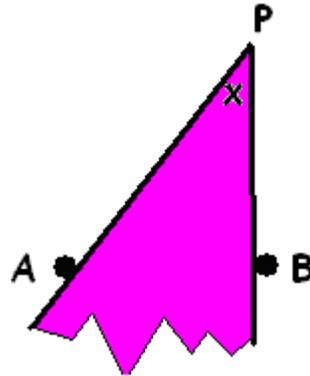
$$h = \sqrt{xy}.$$

HINT: Explain why the two angles marked with a dot are congruent. Use similar triangles to establish $h/x = y/h$.

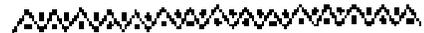
Any mathematics discussion, including the ones each day in the classroom, should really attend to a story, with any one question and its resolution naturally pulling to a natural next question. Mathematics perpetually unfolds.

So what is a natural next question to this discussion on Thales' Theorem and its converse? One immediately comes to my mind ... Why stick with 90 degrees?

Cut a piece of paper so that its corner angle is not 90 degrees. What curve is traced by this corner if it is pushed up between two tacks on a wall at different angles?



Watch your entire unit on circle theorems unfold before your very eyes with your students!



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