TANTON'S TAKE ON …

WHY NEGATIVE TIMES NEGATIVE POSITIVE

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

OCTOBER 2012

I remember asking my fifth-grade teacher why negative times negative is positive. She answered: "It just is. Go back and finish your worksheet." (I must have caught her at a bad moment!) It is a perennial question and one that can be mighty tricky to answer satisfactorily – be it for a young student, for a high-school student, or for ourselves! How might you personally explain to a colleague why negative times negative should be positive?

Some people equate "negative" with "opposite" and state: *taking the opposite of the opposite clearly brings you back to start.* Along these lines I've had teachers describe walks on a number line in left and right directions or talk of credit and debt. ("Suppose you owe someone a debt of \$5. Think about what that might mean.") But this thought only helps illustrates why $--a$ is the same as $+a$. This is a different question! The "opposite of the opposite" does not address why $(-2)\times(-3)$, for example, should be <u>positive</u> six.

> **An idea:** Many folk say that $-a$ is the same as $(-1)\times a$. [Is this obvious?] In which case: $--a=(-1)\times(-1)\times a$. So maybe believing $--a$ equals a is enough to logically deduce that $(-1)\times(-1)$ must equal <u>positive</u> one. Hmm. What do you think?

Negative numbers caused scholars much woe in the history of mathematics. A negative quantity is abstract and giving meaning to an arithmetic of negatives was troublesome. (What does it mean to multiply together a five dollar debt and a two dollar debt?)

"Number" begins in our developing minds as a concrete concept. "5" counts something real - cows in a field, flavors of ice-cream, misplacements of an umlaut in a badly written German sentence. We have youngsters count counters or dots and develop a sense of arithmetic from the tangible. For instance, here is a picture of a sum:

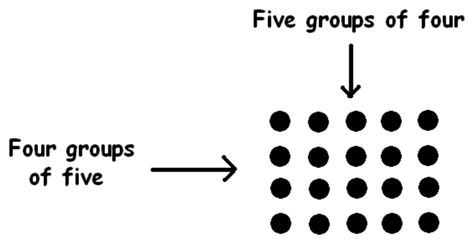
$$2+3=5$$

Looking at the same picture backwards we see $3+2$, and this too must be 5. (It is the same picture!) For any two counting numbers we thus come to believe: $a + b = b + a$.

At this level, multiplication is often seen as repeated addition:

$$\begin{aligned} 4 \times 5 &= \text{ four groups of five} \\ &= 5 + 5 + 5 + 5 \end{aligned}$$

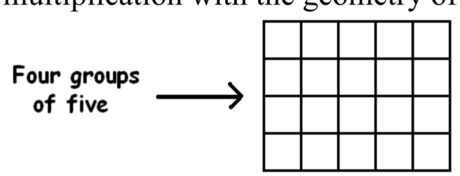
There is no reason to believe that four groups of five should be the same as five groups of four—at least not until the inspiration comes to arrange those four groups of five in a rectangle:



We are then led to believe that $a \times b = b \times a$ for all counting numbers.

As five groups of nothing still gives us very little, we come to say that $a \times 0 = 0$ for all counting numbers a .

If instead of counting dots we count unit squares, we start to associate multiplication with the geometry of area.



Area of 4x5 rectangle

And by dividing a rectangle into pieces we discover a whole host of distributive rules we feel should hold for counting numbers.

	c	d	e
a	ac	ad	ae
b	bc	bd	be

$(a+b)(c+d+e) = ac + ad + ae + bc + bd + be$

COMMIENT: The geometric model for multiplication lets us compute multi-digit products with ease: just chop up a rectangle!

Here is 37×23 .

	20	3
30	600	90
7	140	21

37x23 = 600 + 140 + 90 + 21 = 851

(I can almost do this in my head!)

FOIL or FLIO or LIFO or OILF?
Let go of "first, outer, inner, last"! Have students simply draw rectangles whenever they need to expand parentheses.

	x	y
x	x^2	xy
y	yx	y^2

$(x+y)^2 = x^2 + 2xy + y^2$

The order in which you add individual pieces does not matter. (Let them OILF!)

Also, with rectangles, students can expand expressions for which FOIL is of no help.

	x	a	w	2
a	ax	a^2	aw	2a
b	bx	ab	bw	2b
7	7x	7a	7w	14

$(x+a+w+2)(a+b+7) = ax + a^2 + 7w + \ldots$

So in the realm of counting numbers, the positive whole numbers, we come to believe the following rules of arithmetic:

For all counting numbers a, b and c we have:

1. $a+b=b+a$
2. $a\times b=b\times a$
3. $a\times 0=0$
4. $a(b+c)=ab+ac$

   (along with other variations of the distributive rule).

There are other rules too we can discuss, (that $a+0=0$ and $1\times a=a$, for all counting numbers a, for instance) but the ones listed above are the key players in the issue of this essay.

ENTER …
THE NEGATIVE NUMBERS

Whatever model one uses to introduce negative numbers to students, we all like to believe that $-a$ is a number, which when added to a, gives zero:

5. $a+(-a)=0$

(Whether such a number exists can be debated!) As soon as we start playing the negative number game we are faced with a fundamental question:

Do we feel that the rules of arithmetic listed above are natural and right? Do they feel so natural and so right that we think they should hold NO MATTER WHAT, for ALL types of numbers?

If the answer to this question is YES, then you have no logical choice but to accept negative times negative is positive! Read on!

Comment: Accepting these rules of arithmetic as valid for negative numbers is a choice! You could decide, for instance, that negatives follow a different set of rules. And doing so is fine – you then will be on the exciting path of creating a new type of number system! (And alternative arithmetics can be useful and real. For example, the founders of quantum mechanics discovered that the arithmetic describing fundamental particles fails to follow the rule $a\times b=b\times a$!)

Positive times Negative:

Why should $2\times(-3)$ equal <u>negative</u> six?

If we follow the "repeated addition" model for multiplication, we argue:

$$2\times(-3)=\text{two groups of }(-3)$$
$$=(-3)+(-3)=-6.$$

But at some point we must set multiplication free of its alleged binds to addition and let it be its own independent mathematical operation. ("Repeated addition" has little meaning for $\sqrt{2}\times a$? What are square root two groups of a quantity?) We need the rules of arithmetic—at least the ones we're choosing to believe—to explain why results must be true.

Here's how to examine $2\times(-3)$ in this light. We use rules 3, 4, and 5.

By rule 3 we believe:

$$2\times 0=0.$$

Rewriting zero in a clever way (via rule 5) we must then also believe:

$$2\times(3+(-3))=0$$

Expanding via rule 4 gives:

$$2\times 3\ +\ 2\times(-3)=0$$

We can handle 2×3*, it is* 6*. So we have:*

$$6\ +\ 2\times(-3)=0$$

. *Six plus a quantity is zero. That quantity must be* -6*. Voila!* $2\times(-3)$ *is* -6.

<u>Note</u>: We are not locked into the specific numbers 2 and -3 for this argument.

Negative times Positive:
What should $\left(-2\right)\times 3$ be?

We must now, for sure, let go of our repeated addition thinking! (Does "negative two groups of three" make sense?)

However, by rule 2 we have agreed that $\left(-2\right)\times 3$ should equal $3\times\left(-2\right)$, which we calculate to be -6 by the previous argument. Done!

$\left(-2\right)\times 3$ and all its kin are now under control.

THE BIGGIE: Negative times Negative!

Let's think about $\left(-2\right)\times\left(-3\right)$. Can we follow essentially the same argument as before? Let's try.

By rule 3 we believe:

$$\left(-2\right)\times 0 = 0.$$

Rewriting zero (via rule 5) we must then also believe:

$$\left(-2\right)\times\left(3+\left(-3\right)\right) = 0$$

Expanding via rule 4 gives:

$$\left(-2\right)\times 3 \;+\; \left(-2\right)\times\left(-3\right) = 0$$

We've just done $\left(-2\right)\times 3$*, it is* -6.

So we have:

$$\left(-6\right) \;+\; \left(-2\right)\times\left(-3\right) = 0.$$

. -6 *plus a mysterious quantity is zero. That quantity must be* $+6$. *That is*

$$\left(-2\right)\times\left(-3\right) \textit{ must equal } +6.$$

No choice!

This does it. Rules 3, 4, and 5 simply force negative times negative to be positive!

EXERCISE: Repeat this argument to show that $(-4)\times(-5)$ must be +20.

AN ANSWER TO SATISFY A FIFTH-GRADE JAMES TANTON

Our work here shows the true reason why negative times negative is positive—it is a logical consequence of the axioms we choose to accept in standard arithmetic. I do not, however, recommend offering this abstract rationale to a young student asking the question! But there is an accessible answer that I do think would have satisfied a young James Tanton. It does follow the same rationale as before, but the axioms and logical steps are hidden "behind the scenes." We return to the area model of multiplication.

To examine $\left(-2\right)\times\left(-3\right)$, say, start by looking at 18×17. By the rectangle method we have:

	10	7
10	100	70
8	80	56

18x17 = 100 + 80 + 70 + 56 = 306

Now let's have some fun. Instead of thinking of 17 as $10+7$, think of it as $20-3$! Do we still get 306? We do!

	20	-3
10	200	-30
8	160	-24

18x17 = 200+160-30-24 = 306!!

[It is fine here to go back to our early thinking: $10\times\left(-3\right) =$ ten groups of $\left(-3\right)$.]

How about having fun with 18 instead? Think of it as $20-2$.

	10	7
20	200	140
-2	-20	-14

18x17 = 200-20+140-14 = 306!!

Yep! 306.

[I find students will naturally choose to calculate $(-2)\times 10$ as $10\times(-2)$. But I do ask: "Do you think it is okay to think of it this way?"]

Okay … The answer is 306 no matter how we seem to look at it. So let's look at it having fun with both numbers at the same time! Do you see that this now contains the mysterious $(-2)\times(-3)$?

	20	-3
20	400	-60
-2	-40	??

18x17 = 400-60-40+?? = 306

We know the answer is 306. So what must the value of the question marks be? Well … the math is telling us $400-60-40+??=306$.
That is, $300+??=306$. We must be dealing with the value $+6$.

That is, the math is saying it wants $(-2)\times(-3)=+6$. Negative times negative is positive.

EXERCISE: Repeat this argument to show that $(-4)\times(-5)$ must be +20. (Perhaps look at 36×25 or … ?)

For the VERY Technically Minded: Professional mathematicians reading this essay will be quick to point out that the arguments I have presented still contain unspoken assumptions. For example, when presented with

$$(-6)+??=0$$

we assumed that the question marks must have the value 6. How do we know this? The answer might be to add 6 to the left side and to the right side and to make use of the associative rule (not mentioned in this essay).

$$6+((-6)+??)=6+0$$

$$(6+(-6))+??=6+0$$

Rule 5 allows us to conclude $0+??=6+0$. The "property of zero" (only quickly mentioned in this piece) then gives: $??=6$.

CHALLENGE 1: *Identify ALL the rules of arithmetic one should list in order to properly conduct the arguments listed in this newsletter!* (Mathematicians call them the "axioms of a ring.")

CHALLENGE 2: P*rove that every number a has only ONE additive inverse. That is, if $a+b=0$ and $a+c=0$ also, then b and c must be the same number.* (Only after establishing this will mathematicians say that the "additive inverse" $-a$ of number a is a "well defined" concept.)

CHALLENGE 3: *Prove that $-a$ and $(-1)\times a$ are the same number.*[HINT: $a+(-1)\times a=1\times a+(-1)\times a=\cdots$.]

CHALLENGE 4: *Prove that $-(-a)$ and a are the same number.*

CHALLENGE 5: Prove $-0=0$.