TANTON'S TAKE ON …

NEGATIVE NUMBER ARITHMETIC

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

JUNE 2013

The following appears in Chapter 4 of:

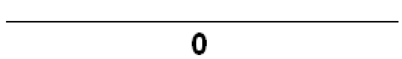
THINKING MATHEMATICS! Vol 1
Arithmetic = Gateway to All

available at www.lulu.com .

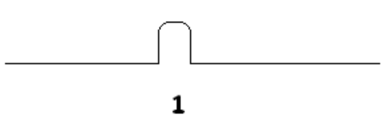
Despite its almost insulting simplicity, the model described here provides a very powerful tool for understanding negative numbers.

We begin with a story that isn't true.

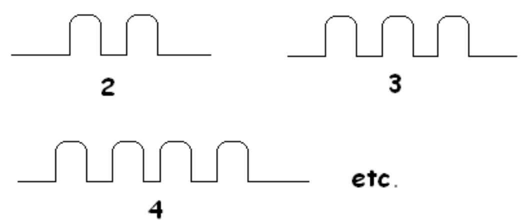
When I was a young child I spent my days sitting in a sandbox at the back of my yard (not true). And being a very serene child I took the time each morning to level the sand in my box and make a perfectly flat horizontal surface. (Also not true.) This very much appealed to my tranquil sensibilities, so much so that I decided to give this level state a name. I called it "zero."



I spent many an hour admiring my zero state. (Still not true.) But then one day I had an epiphany! I realized I could reach behind where I was sitting, grab a handful of sand and make a pile. And I called the one pile the "1" state.

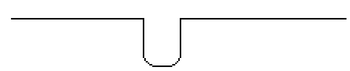


And with more epiphanies, I discovered two piles, three piles, and so on, which I called "2" and "3," etc.

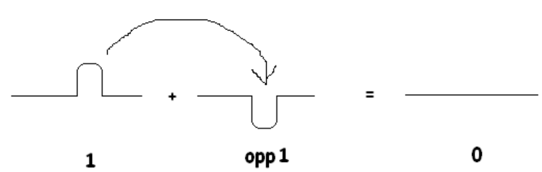


Hours of mathematical fun were had as I discovered the counting numbers through piles of sand.

BUT THEN … one day I had the most astounding epiphany of all! Instead of using a handful of sand to make a pile I realized I could take away a handful of sand and make the OPPOSITE of a pile, namely, a hole!



I called this "*opp* 1" for the "opposite of one pile." And notice that "opposite," in some sense, is really the right word because a hole a pile "level out" to bring me back to zero.



$$1 + opp1 = 0$$

In the same way I wrote "*opp* 2" to represent the opposite of two piles, namely, two holes, and "*opp* 5" for the opposite of five piles, namely, five holes. And so on.

Practice: Draw a picture for $2 + opp\ 2$ and draw the effect of filling holes with piles.

Question:

a) JinSe says that $3 + opp\, 2$ equals one pile? Is she correct? Draw a picture.

b) Harold says $5 + opp\, 7$ equals 2 holes. Is he correct? Draw a picture.

Question:

a) What is the opposite of a hole?

b) What is the opposite of the opposite of the opposite of three piles?

c) What is the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of ten hundred holes?

WHAT THE REST OF THE WORLD WRITES AND THINKS

As a young child I had, allegedly, discovered a whole host of new numbers, the opposites of the counting numbers. The world calls these the negative numbers. And instead of writing *opp* for opposite, people use a tiny dash "-".

So …

2 = two piles

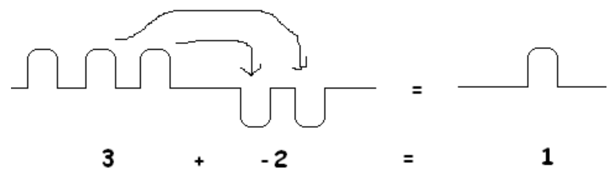
-2 = opposite of two piles
= two holes

--2 = opp opp 2 = two piles

-1 = one hole

-----------7 = seven holes

Unfortunately, notational matters become a little confusing when one starts combining piles and holes. For example, three piles and two holes together, giving one pile, is written $3 + -2 = 1$ in this notation.
($3 + opp\, 2 = 1$ is better!)



And $5 + -7 = -2$ is a clunky way of writing "five piles and seven holes is equivalent to two holes."

Question: a) Ali invented the notation "$4P + 5H + 3H + 2P = 2H$". What do you think he means by this? Do you like his notation?

b) Cuthbert writes $\bar{3} + \underline{4} + \bar{5} = \bar{4}$. What do you think he means by his notation?

Practice: The statement:

$$-3 + 7 = 4$$

reads "3 holes and 7 piles makes 4 piles," and the statement:

$$17 + -6 + -4 + 6 + -20 = -7$$

reads "17 piles and 6 holes and 4 holes and 6 piles and 20 holes makes 7 holes."

Translate each of the following, and give the numerical results:

a) $5 + -9 + 2$
b) $3 + -10 + 11 - 5$
c) $2 + -2 + 2 + -2 + 2 + -2$
d) $-6 + -1 + -2 + -3$

Question: Pandi writes:

$$----5 + --2 + -------8$$

This actually makes sense! What does it mean and what is the answer?

Many educators will not allow students to write "plus minus" in their work.

$$5 + -7 = -2$$

This is because "$-$" is seen as an operator on equal par with "$+$." They call it "subtraction." And in this context the appearance of two consecutive operator symbols in an expression is wrong.

But, philosophically, there is no "new" operation here. **Subtraction is the <u>addition</u> of the opposite.**

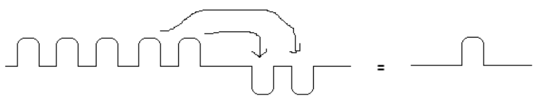
For example,

$$5-2$$

seen as "five take away two" is really

$$5+-2.$$

"Five piles plus the ADDITION of two holes," giving three piles.



5 + -2 = 3

(One way to "take away" piles is to add holes that will annihilate them!)

In the same way:

$$6-4+1$$

"six take away four plus one," is really:

$$6+-4+1$$

"Six piles PLUS four holes PLUS one pile" giving three piles.

Question: Young children are usually taught to think "subtraction" and are often asked to practice examples like these:

$$4-1=??$$

$$10-7=??$$

$$8-5=??$$

But is it natural for children to enquire about problems of the type:

$$3-5=??$$

$$1-4=??$$

$$66-103=??$$

Sometimes they are told these problems do not have answers.

Do you think if young children were taught about piles and holes they could understand answers to questions like these?

Question: Should -0 equal 0?

GROUPS OF PILES AND HOLES

Here's a vague question:

What is the opposite of three piles and two holes?

Is this asking for

The opposite of three piles AND THEN the addition of two holes

or

The opposite of both three piles and of two holes?

Question: One of these interpretations has the answer "five holes" and the other interpretation the answer "one hole." Which answer belongs to which interpretation?

Mathematicians use parentheses to help clarify such confusion. Parentheses group objects together.

EXAMPLE: Interpret $-(3+-2)$.

Answer: This means the opposite of everything in the parentheses, the opposite of BOTH three piles and two holes. And what is the opposite of both three piles and two holes? Clearly, three holes and two piles!

$$-(3+-2)=-3+2$$

This is one hole.

$$-(3+-2)=-3+2=-1$$

Comment: Most people write

$$-(3-2)=-3+2=-1.$$

Do you see the slight difference?

EXAMPLE: Interpret $-(3+-4+2)+5$.

Answer: This is the opposite of all of 3 piles, 4 holes and 2 piles, AND THEN the addition of 5 piles. So we have 3 holes, 4 piles, 2 holes, and 5 piles.

$$-(3+-4+2)+5=-3+4+-2+5$$

Comment: Most write this as

$$-(3-4+2)+5=-3+4-2+5.$$

EXAMPLE: Interpret $6-(5-2)$.

Answer: This is six piles and the opposite of both 5 piles and 2 holes.

$$6-(5-2)=6+-5+2$$

This equals 3 piles.

EXAMPLE: Interpret $(5-2+1)-(3-2)$.

Answer: This is 5 piles, 2 holes and 1 pile all grouped together PLUS the opposite of both 3 piles and 2 holes.

$$(5-2+1)-(3-2)=5+-2+1+-3+2$$

This equals 3 piles.

BRINGING THIS UP TO ALGEBRA SPEED WITH NO FUSS

EXAMPLE: Interpret $-(x-y)$.

Answer: This is the opposite of x piles AND y holes. The answer is obviously x holes and y piles.

$$-(x-y)=-x+y.$$

EXAMPLE: Interpret $-(10-T+7-3+a)$.

Answer:
$-(10-T+7-3+a)=-10+T-7+3-a$

Comment: Say this out loud and say something rude! "The opposite of 10 piles AND T holes AND 7 piles AND 3 holes AND a piles is …"

EXAMPLE: Rewrite $(5-w)-(2-w)$.

Answer: (5 piles and w holes) with the opposite (2 piles and w holes) is:

5 piles, w holes, 2 holes and w piles.

I see with a picture in my mind this is just three piles.

$$(5-w)-(2-w)=3.$$

EXERCISE: What's $-x$ if:

a) x is seven piles?
b) x is seven holes?
c) x is 16?
d) x is -16?

Practice: Explain why $2-(20-x)$ is just x piles and 18 holes. How is that normally written?

PEDAGOGICAL COMMENT

In writing $-(a-b)$ as $-a+b$, for example, we have just performed what many educators call **distributing the negative sign**. Many a high-school student struggles with practice. But I find if I just spend half a class period giving the story of piles and holes, this worry dissipates. And if a student does struggle with an equation like this, I advise "Just read it as piles and holes."

CARD PILE ADVENTURE

Here is a curious little puzzle. The algebra of negative numbers is needed to properly explain what we see.

a) Take 10 red cards and 10 black cards from a deck of cards. Shuffle your 20 cards and arbitrarily split them into two equal piles. Count the number of red cards in the left pile and the number of black cards in the right pile. What do you notice? Repeat this activity a few more times.

b) Shuffle your 20 cards and this time split them into a pile of 6 and a pile of 14 cards. Count the number of red cards in the small pile and count the number of black cards in the large pile. Take the (positive) difference of those two counts. Did you get 4? Repeat this a few more times.

c) Shuffle the 20 cards again and this time split them into a pile of 9 cards and a pile of 11 cards. Count the number of red cards in the small pile, count the number of black cards in the large pile and take the (positive) difference of this count. What did you get? Repeat a few more times. What do you notice?

d) Complete the following table:

Small Pile	Large Pile	Difference: R in small & B in large
10	10	0
9	11	
8	12	
7	13	
6	14	4
5	15	
4	16	
3	17	
2	18	
1	19	
0	20	

Any patterns?

e) Suppose, in a game with 5 cards in the small pile and 15 cards in the large pile, I counted three red cards in the small pile. Complete the following table:

	Small Pile 5	Large Pile 15
# reds	3	
# blacks		

What is the difference of counts of red cards in the small pile and black cards in the large pile?

f) Suppose the small pile has P_1 cards and the large pile has P_2 cards (hence $P_1 + P_2 = 20$). Suppose it turns out there are R red cards in the small pile. Complete the following table as an abstract exercise:

	Small Pile P_1	Large Pile P_2
# reds	R	
# blacks		

What can you say about the difference between the number of red cards in the small pile and the number of black cards in the large pile?

A CLASSIC: MILK AND SODA

Saku has a glass of soda and a glass of milk. She takes a tablespoon of soda from the first glass and haphazardly stirs it into the milk. She then takes a tablespoon of the milk/soda mixture and transfers it to the soda. Both drinks are now "contaminated."

Here's the question: Which has more foreign substance? Is there more foreign milk in the soda than foreign soda in the milk? Is it the other way round? Or is it impossible to say?

COMMENT: Part a) of the previous activity can be thought of as a "ten molecule" version of this puzzle.

See

www.jamestanton.com/?p=991

for a video on this milk and soda puzzle.

WHY IS NEGATIVE TIMES NEGATIVE POSITIVE?

Glad you asked!

This is the subject of the OCTOBER 2012 essay! You will find a copy of it at

www.jamestanton.com/?p=1072.

All the details resolving this thorny question are there.

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