TANTON'S TAKE ON …
"DOES ORDER MATTER?"

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

JUNE 2012

If I were Emperor of the Math Teaching Universe I would proclaim three edicts. One would be the complete eradication of the words "permutation" and "combination" from all school text books. And with that naturally comes a ban on the question: "Does order matter?" This question has induced far too many headaches for student and teacher alike over the decades. The pain has to stop!

Here is Tanton's three-step approach to pain-free understanding of counting problems. There is no difference between a permutation and a combination and I wish the world would realize this.

Comment: What appears here is a swift overview of ideas for you as a teacher. With students I give more lead-in examples and invite much discussion. See www.jamestanton.com/?p=659 for downloadable materials to share with algebra students. For a more advanced set of notes, see Chapter 16 of *THINKING MATHEMATICS! Vol 2: Advanced Counting and Advanced Algebra Systems* available at www.lulu.com.
(Chapter 17 is really good too!)

[**Aside:** Didn't I say three edicts?]

STEP ONE:

THE MULTIPLICATION PRINCIPLE: If there are a ways to complete a first task and b ways to complete a second, and no outcome from the first task in any way affects a choice of outcome for the second, then there are $a \times b$ ways to complete both tasks as a pair.

This simple idea is sometimes called the counting principle. It shows, for example, that if there are 4 possible routes from A to B, and 5 possible routes from B to C, then there are 20 ways to travel from A to C via B.

The principle also extends to more than two tasks. For example, I own three different shirts, five different pairs of trousers, and two different pairs of shoes. How many different outfits do I possess? Answer: $3 \times 5 \times 2 = 30$. (I am set for a month!)

[The caveat in the statement of the principle is important. If, for example, I will never wear my lilac shirt with my mushroom-pink trousers, then the computation $3 \times 5 \times 2$ does not apply.]

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STEP TWO:
REARRANGING LETTERS

My name is JIM. In how many ways can I arrange the letters of my name?

Answer 1: A "brute force" approach would simply list all possibilities:

JIM	MJI	IMJ
JMI	MIJ	IJM

There are six ways to arrange the letters.

Answer 2: Use the multiplication principle: We have three slots to fill:

___ ___ ___

The first task is to fill the first slot with a letter. There are 3 choices of letter to use and so 3 ways to complete this task.

The second task is to fill the second slot. There are 2 ways to complete this task (as one letter has already been used).

The third task is to fill the third slot. There will be only 1 way to complete this task.

3 2 1

By the multiplication principle, there are $3\times 2\times 1=6$ possibilities.

Actually, the truth is I am shifting to "James" for my name (and confusing many in the process). How many ways can I arrange the letters of JAMES?

Answer: As a series of five tasks we see there are $5\times 4\times 3\times 2\times 1=120$ arrangements.

This game naturally motivates:

Definition: The product of the integers from 1 to N is called N *factorial* and is denoted $N!$.

These numbers grow very large very quickly:

$1!=1$

$2!=2\times 1=2$

$3!=3\times 2\times 1=6$

$4!=4\times 3\times 2\times 1=24$

$5!=5\times 4\times 3\times 2\times 1=120$

$6!=6\times 5\times 4\times 3\times 2\times 1=720$

$7!=7\times 6\times 5\times 4\times 3\times 2\times 1=5040$

$8!=8\times 7\times 6\times 5\times 4\times 3\times 2\times 1=40320$

What is the first factorial larger than billion? What is the largest factorial your calculator can handle?

Historical Comment: In 1729, at the age of 22, Swiss scholar Leonhard Euler found the formula for a general function that matches the factorials. It gives a curve that "connects the dots" for a graph of the factorial values. He called his function the *Gamma Function* and it allows us to plug in non-integral values for factorials. Euler discovered, for instance, that ½! equals $\sqrt{\pi}/2$. Very strange!

Aside: *In how many ways can one rearrange the letters of the word FACETIOUSLY?* This question is just an excuse to ask an interesting non-math question: What do you notice about the vowels (including *y*) of *facetiously*? As far as I am aware, there is only one other word in the English language with this property. What is it? (Email me if I am wrong and you can find two!)

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So far all is fine and dandy. But we are lucky that my name isn't BOB.

We need to handle repeated letters!

In how many ways can one arrange the letters BOB?

Here's a labored, but revealing, way to think about this problem.

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If the B s were distinguishable – written, say, as B_1 and B_2 - then the problem is easy to answer: There are $3! = 6$ ways to rearrange the letters B_1OB_2 . The list is:

$$B_1OB_2 \quad B_1B_2O \quad OB_1B_2$$
$$B_2OB_1 \quad B_2B_1O \quad OB_2B_1$$

But if the B s are no longer distinguishable, pairs in this list of answers "collapse" to give the same arrangement.

$$BOB \quad BBO \quad OBB$$

We must alter our answer by a factor of two and so the number of arrangements of BOB is $\frac{3!}{2} = 3$.

How many ways are there to rearrange the letters of the word CHEESE?

Answer: If the three Es are distinct – written E_1, E_2, and E_3, say – then there are $6!$ ways to rearrange the letters $CHE_1E_2S\,E_3$. But the three Es can be rearranged $3! = 6$ different ways within any one particular arrangement. These six arrangements are seen as the same if the Es are no longer distinct:

$$HE_1E_2SCE_3 \quad HE_3E_1SCE_2$$
$$HE_1E_3SCE_2 \quad HE_3E_2SCE_1 \quad \rightarrow \quad HEESCE$$
$$HE_2E_1SCE_3 \quad HE_2E_3SCE_1$$

We must divide our answer of $6!$ by $3!$ to account for the groupings of six that become identical. There are thus $\frac{6!}{3!} = 120$ arrangements of CHEESE.

Comment: For BOB, the "2" in the denominator is actually $2!$.

Explain why the number of ways to arrange the letters of CHEESES is $\frac{7!}{3!2!}$.

Aside: *In how many ways can one arrange the letters of BOOKKEEPER?* As far as I am aware bookkeeper (ignoring its variants "bookkeeping," etc) is the only word in the English language with three consecutive sets of double letters!

Fun Question: How many ways can you rearrange the letters of your full name? (Will you include spaces?)

Consider CHEESIESTESSNESS, the quality of being the cheesiest of cheeses! We are at the point now where we can readily see that there $16!/(5!6!)$ ways to rearrange its letters. But it is actually better to write this answer as:

$$\frac{16!}{1!1!5!6!1!1!1!}$$

$1!$ for the one letter C
$1!$ for the one letter H
$5!$ for the five letters E
$6!$ for the four letters S
$1!$ for the one letter I
$1!$ for the one letter T
$1!$ for the one letter N

This offers a self check: The numbers appearing on the bottom should sum to the number appearing on the top.

Strange Question: There are no Ps in the word. Should we also include $0!$, whatever it means, in the denominator?

STEP 3: THE LABELING PRINCIPLE

How many ways to arrange the letters of the Swedish pop group name ABBA?

Answer: $4!/(2!2!) = 24/4 = 6$.

How many ways can we rearrange the letters of AABBBBA?

Answer: $7!/(3!4!)$

How many ways can we rearrange the letters of AAABBBBCCCCCC?

Answer: $\frac{13!}{3!4!6!}$

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Let's look at this third problem and phrase it in a different way:

> Mean Mr. Muckins has a class of 13 students. He has decided to call three A students, four B students, and six C students. In how many ways could he assign these labels?

Answer: Let's imagine all thirteen are in a line. Here's one way he can assign labels:

A C B B B A C C C C A C B

Here's another way:

B A C C B C C C B A C B A

and so on. We see that this labeling problem is just the same problem as rearranging letters. The answer must be $\frac{13!}{3!4!6!} = 60,060$.

> Of 10 people in an office 4 are needed for a committee. How many ways?

Answer: Imagine the 10 people standing in a line. We need to give out labels. Four people will be called "ON" and six people will be called "LUCKY." Here is one way to assign those labels:

L L O O L O L L L O

We see that this is just an arrangement problem. The answer is: $\frac{10!}{4!6!} = 210$

In general, we have …

> **THE LABELING PRINCIPLE**
> Each of N distinct objects is to be given a label.
> If a of them are to have label "1," b of them to have label "2," and so on, then the total number of ways to assign labels is:
>
> $$\frac{N!}{a!b!\cdots z!}$$

This is the key principle behind everything! I admit it is a bit of work getting to this point, but in my classroom experience it has always been fun and joyful time for my students going through these rearrangement games.

Do make the work up to this point fun and playful. Choose silly words to rearrange. Make it a bit of contest as to who has the most rearrangements to their name. (Challenge kids to actually work out the numbers!) Good, true understanding results!

PUTTING THE LABELING PRINCIPLE TO USE:

> 1. *Three people from a group of twelve are needed for a team. In how many different ways can a team be formed?*

Answer: The 12 folk are to be labeled: 3 as "on the team" and 9 as "off the team." The answer must be $\frac{12!}{3!9!} = 220$.

> 2. *Fifteen horses run a race. How many possibilities are there for first, second, and third place?*

Answer: 1 horse will be labeled "first," 1 will be labeled "second," 1 "third," and 12 will be labeled "losers." The answer must be: $\frac{15!}{1!\,1!1!12!}$.

Comment: Be sure to assign a label to each and every person/object in the problem. This fits the "self check" we described earlier.

Another comment: The first problem above is called a "combination" problem and the second a "permutation" problem. How confusing! They are both simply labeling problems!

What might folk call these next two problems? PermuCombo- nightmares??

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3. A "feel good" running race has 20 participants. Three will be deemed equal "first place winners," five will be deemed "equal second place winners," and the rest will be deemed "equal third place winners." How many different outcomes can occur?

Answer: Easy! $\frac{20!}{3!\,5!12!}$.

4. From an office of 20 people, two committees are needed. The first committee shall have 7 members, one of which shall be the chair and one the treasurer. The second committee shall have 8 members. This committee will have 3 co-chairs and 2 co-secretaries and one treasurer. In how many ways can this be done?

Answer:
Keep track of the labels. Here they are:

1 person will be labeled "chair of first committee"
1 "treasure of first committee"
5 "ordinary members of first committee"
3 "co-chairs of second committee"
2 "co-secretaries of second committee"
1 "treasurer of second committee"
2 "ordinary members of the second committee"
5 people will be labeled "lucky," they are on neither committee.

The total number of possibilities is thus: $\frac{20!}{1!1!5!3!2!1!2!5!}$. Piece of cake!

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For those worried by the explicit lack of combinations and permutations...

5 people are to be chosen from 12 and the order in which folk are chosen is not important. How many ways?

Answer: 5 people will be labeled "chosen" and 7 "not chosen. There are $\frac{12!}{5!7!}$ ways to accomplish this task.

And this is the classic $_{12}C_5$ formula.

5 people to be chosen from 12. The order in which they are chosen is considered important. How many ways?

Answer: We have:
1 person labeled "first"
1 person labeled "second"
1 person labeled "third"
1 person labeled "fourth"
1 person labeled "fifth"
7 people labeled "not chosen"

This can be done $\frac{12!}{1!1!1!1!1!7!}$ ways.

And this is the classic $_{12}P_5$ formula.

7 men and 6 women in an office. How many ways are there to make a committee of five if the committee must consist of 2 men and 3 women?

This is "Multi-Stage Labeling."
(More on this in my student notes.)

TASK 1: Deal with the men: $\frac{7!}{2!5!}$ ways.

TASK 2: Now women: $\frac{6!}{3!3!}$ ways.

By the multiplication principle there are $\frac{7!}{2!5!}\times\frac{6!}{3!3!} = 21\times 20 = 420$ ways to complete both tasks.

CHALLENGE FOR YOU! Examine your current textbook and show that every permutation/combination formula, every example, and every question in it is simplified considerably when phrased in terms of labels!

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