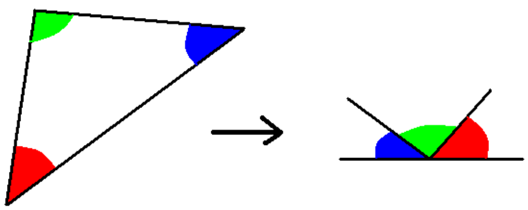
TANTON'S TAKE ON …

"180^0 IN A TRIANGLE"

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

JULY 2012

We are all taught/told that the three interior angles of a triangle sum to half a turn, 180°.



And we might test this claim by cutting a triangle from a piece of paper, tearing off its three corners and aligning them as the right portion of the picture suggests: The three angles, more or less, line up along a straight line.

When first showed this paper-tearing verification I worried about two things.
1. We never actually see a perfect straight angle. Most people argue this is due to human imperfection – we can't drew perfect straight-edged triangles nor cut perfectly along straight lines. But those folk have decided *a priori* that the result should be true. What then made them decide the angles have this property in the first place?
2. Even if we could draw and cut triangles to a level of precision that would satisfy the human eye, how do we know that three angles summing to 180° is not just a local effect? For example, all physical demonstrations in my neighborhood give credence to the idea that the Earth is absolutely flat. That is my local experience. When we test the angles in triangles we only ever seem to draw triangles the sizes of pieces of paper. Have we ever tested the angles in a triangle the size of the solar system? How about one the size of an atom? I can discover that the Earth is not flat if I go to an extreme scale. Why don't we ever go to extreme scales to test triangles?

I've taught multiple sections of ninth and tenth grade geometry each and every year of my school teaching career. And I have this discussion with my students. We begin to doubt this standard "fact" about angles in triangles.

At this point I reveal one exercise that does seem to suggest, after all, that angles in triangles do sum to half a turn. I call it the PENCIL TURNING TRICK. Although in class we apply it to a specific triangle drawn on a white-board, it is really an exercise of the mind: we can see that the trick will hold for any triangle, even one the size of the solar system. It seems to be a universal argument that goes beyond tearing off the corners of any specific triangle.

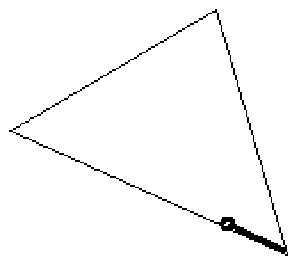
Comment: I am deliberately being coy with my wording. There is a concern/surprise about this trick that will be revealed only at the end of this letter. No peaking ahead!

By The Way … This content appears as chapter 3 of *GEOMETRY: Volume 1* available at www.lulu.com. (Search under TANTON.) I am delighted this book has been adopted by three high-schools.

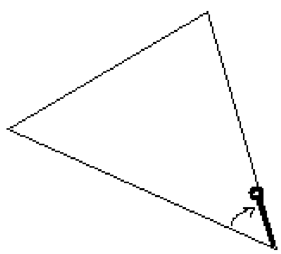
THE PENCIL TURNING TRICK:

Step 1: *Draw a large triangle on a piece of paper or on a white-board, one large enough to contain a small pencil.*

Step 2: *Place a pencil along one side in one corner of the triangle. Take note of the direction in which the pencil is pointing.*

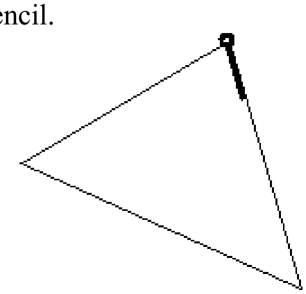


Step 3*: Turn the pencil through the first angle.*

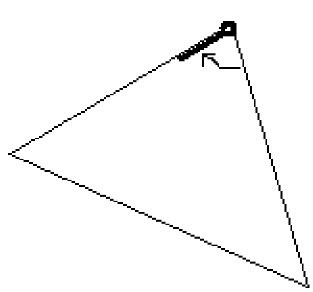


This alters the direction of the pencil by the amount of turning given by this first angle.

Step 4: *Slide the pencil along the side of the triangle.* This does not change the direction of the pencil.

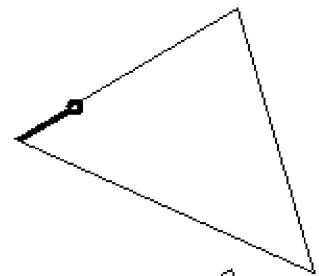


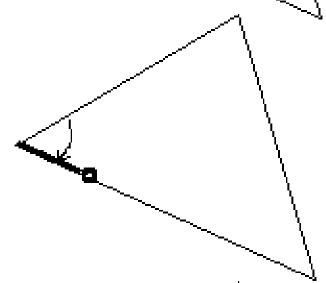
Step 5: *Turn the pencil through the second angle.*

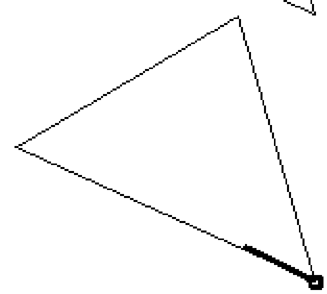


The pencil has now undergone the total amount of turning as specified by two interior angles of the triangle.

Step 6: *Slide the pencil to the third vertex of the triangle and apply the amount of turning given by that angle. Then slide the pencil back to start.*







Compare the final position of the pencil with its initial position to see …

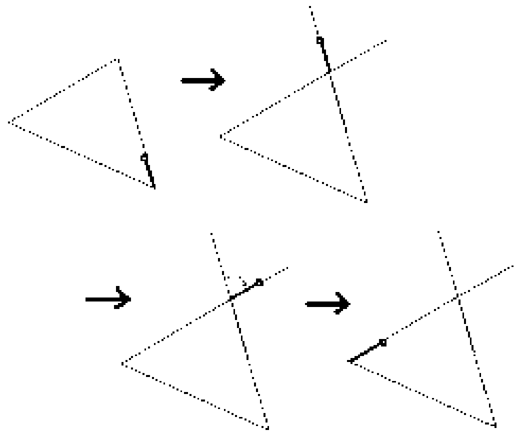
The pencil has undergone one half turn!

That is, the sum effect of turning via the three interior angles of a triangle is 180^{o}.

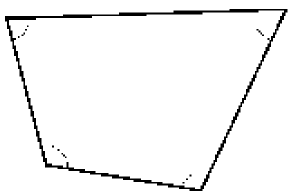
It seems we must conclude:

THE THREE INTERIOR ANGLES OF A TRIANGLE MUST SUM TO 180^{o}.

COMMENT: Some people may object to the fact that we rotated the pencil first about its tip, then its end, and then its tip again. This seems inconsistent. However, with the aid of the vertical angle theorem – the two angles across the vertex of a pair of crossing lines are congruent - we can be consistent within the process and always rotate the pencil about its tip. Simply slide the pencil beyond the top vertex of the triangle and rotate it though the vertical angle. Since this angle has the same measure as the interior angle at this vertex, the overall effect is the same.

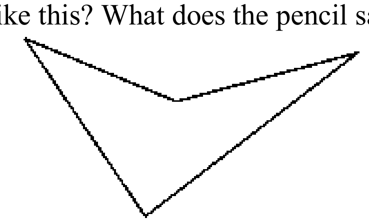


Exploration 1: a) Draw a four-sided figure and apply the same pencil trick to it.



Show that the four interior angles seem to sum to two half-turns, that is, to 360^o.

b) Do you think that the interior angles of <u>all</u> quadrilaterals sum to 360^o, even concave ones like this? What does the pencil say?

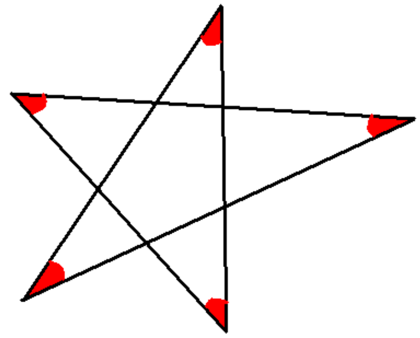


Exploration 2: Explore five-sided figures. What can you say about the sum of the interior angles of any such figure?

(WARNING: A pencil inside a pentagon undergoes more than just a single half turn!)

Explore 9-sided polygons and 14-sided polygons. Care to find what seems to be the general formula for the sum of the interior angles of a polygon with N sides?

Exploration 3: What does the pencil seem to say about the five angles at the tip of a randomly drawn five-point star?



Seven-pointed stars?

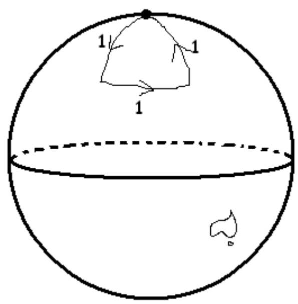
This pencil turning trick is a big hit amongst all my students (and my teachers in geometry curriculum workshops). It seems to make clear how angles in polygons behave – without being tied to any specific polygon at hand. **It is a pity we really shouldn't believe anything the pencil suggests!!!**

Here's a classic puzzler that explains the problem:

One day a woman decides to take a three-mile walk. She first heads directly south for one mile, trotting along at a happy pace, admiring the sunshine and the wildlife. Then she turns left and heads directly East for a mile, all the while enjoying the smell of the sweet air and the glorious sights of nature around her. Next, she turns left one more time and heads directly North just for one more mile. Surprisingly, after this third one-mile stretch she finds herself back to where she started!

What colour was the bear she saw on her walk?

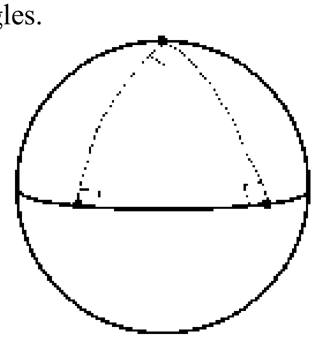
ANSWER: It is possible to walk one mile south, one mile east, and one mile north to end up back at start. …if you start at the North Pole of the Earth!! (Consequently any bear seen must be a polar bear and hence white.)



CHALLENGE: Actually there are other starting positions on surface of the Earth where such a feat can also be accomplished. (There are no bears, alas, at those locations.) Can you think of alternative solutions to the puzzle?

Let's take this further …

Suppose the woman walks south all the way to the equator, turns 90° east and walks a quarter away around the equator and then turns 90° north to end back up at the North Pole. This creates a large triangle with three right angles.



The angles in this triangle <u>do not</u> add to 180°. But here is the problem … **The pencil trick will lie and still say that the angles of this triangle add to half a turn!**

Imagine, starting at the North Pole, the woman pushes the pencil straight in front of her towards the equator. **From her perspective**, she is pushing that pencil along a straight line, always in front of her. (You might have an objection right now, but read on!) When she reaches the equator she rotates the pencil to point east. After pushing the pencil straight (from her perspective) along the equator, she then turns it north. Next, she pushes the pencil straight north until she comes back to the North Pole and rotates it through the final angle. Can you picture in your mind's eye that she will see the pencil back in its original position but pointing in the opposite direction, just as though it had undergone a 180 degree rotation?

What's going on?

Of course you are no doubt arguing that this example does not apply because the Earth is not flat and the lines along which she is pushing the pencil are actually curved (even though she thinks they are straight). Fair enough. But from this example, we can certainly say that the pencil trick will lie for triangles drawn on non-flat surfaces. In particular, it will say "angles in triangles add to 180°" for triangles on spheres, on warped white-boards, on curved pieces of paper, … on anything! … even when we can draw triangles whose angles clearly don't! (the woman's big walk, for example).

So then … How do we know our boards and pieces of paper aren't ever-so-slightly warped? How do we know that the pencil isn't lying for all the examples we've done so far? More disturbingly, how do we know that the pencil isn't lying for "flat" surfaces as well?

We are in a pickle … and welcome to the beginnings of geometry!

There is a way out of this fix. It calls on being very honest about what we <u>choose</u> to believe and being absolutely upfront to the world about the choice. Here goes …

We have currently two tangled ideas. For whatever reason, we want to believe that angles in triangles sum to 180 degrees. We feel that this is how a "flat" space should behave. And the fact that we can form triangles on the Earth whose angles sum to more than 180 degrees shows us that the Earth is not "flat."

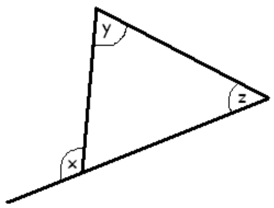
But how do we explain what flat means? <u>Answer</u>: By making what we want to be true its very definition!

DEFINITION: *Geometry is called <u>flat</u> if it is assumed that the three interior angles of any triangle will sum to 180 degrees.*

Geometry at the high-school level is assumed to be flat, and in my geometry course, it is our first fundamental postulate.

FIRST FUNDAMENTAL POSTULATE: *We take it as a given that the interior angles of any triangle sum to* 180^o.

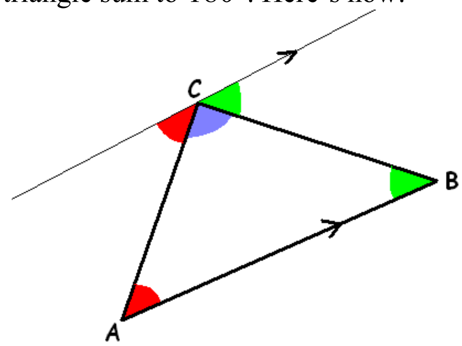
Exercise: For this picture in flat geometry, explain why angle *x* equals the sum of angles *y* and *z*.



Use this picture to explain why the five angles at the tip of a five-pointed star do indeed sum to 180°. (This establishes that the pencil wasn't lying at least for flat five-pointed stars!)

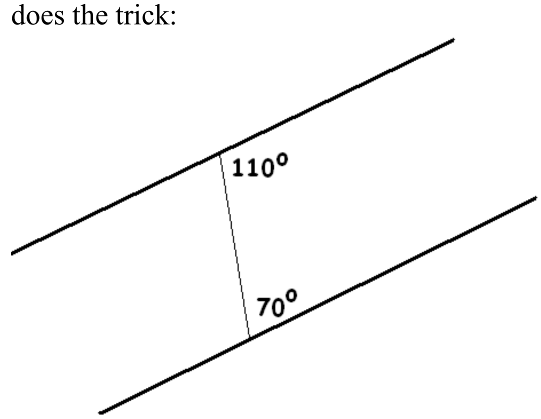
PEDAGOGICAL COMMENT: There are alternative formulations of our "flat" postulate, most notably Euclid's Parallel Postulate. Not to diminish Euclid's brilliance, his parallel postulate can appear to students as a mighty strange and un-natural place to begin making assumptions for a study of geometry. No doubt he formulated this postulate only after many years of thinking deeply about the key features that seem to make geometry work. We don't offer students "many years of thinking deeply" about good choices for a beginning theory of the topic, and instead subject them to the end-result of this process. This, in my opinion, is antithetical to deep and true learning. The question of the sums of angles in a triangle seems to be a very natural and deep mystery for students, one that leads to the same logical beginnings of Euclidean geometry. This approach also brings home the idea—right away—that accepting this postulate is a CHOICE and that there are valid alternative geometries, spherical geometry, for instance, for which this postulate is not true. Euclid's parallel postulate, for students, doesn't provide the same immediate depth of insight.

Of course, Euclid did derive from his parallel postulate the result that the angles in a triangle sum to 180°. Here's how:



Euclid's Proof that the angle in a triangle sum to 180°**:** *Given a triangle* ABC *draw a line through* C *parallel to the base* $\overline{AB}$. (Actually, here is one slip Euclid made – he assumed, without comment, that one can always draw lines through given points parallel to given lines.) *By his parallel postulate, the red angles, being alternate interior angles, are congruent, as are the green angles. It is now readily apparent from the diagram that the three interior angles of the triangle sum to a straight angle.*

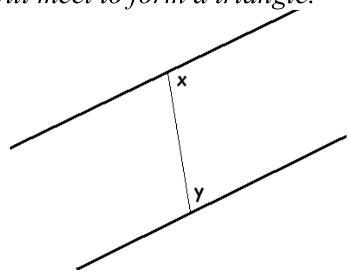
In my geometry course I am, of course, expected to discuss and explore parallelism. Our assumption of flatness naturally leads into it. An exercise such as the following does the trick:



Explain why the bold lines, if extended further to the right, can never meet to form a triangle (in flat geometry at least). Also explain why they can never meet if extended to the left.

This exercise is not Euclid's parallel postulate – although it is clearly closely related.

EUCLID'S PARALLEL POSTULATE in a nutshell: *In this picture, if the measures of angles* x *and* y *sum to less than* 180°, *then the bold lines, if extended to right, will meet to form a triangle.*



Given our work on triangles, this result seems "natural" and right to students (especially if one is specific and says that x has measure 100° and y measure 70°, the added information that the third angle of the triangle formed will be 10° will come).

TOUGH CHALLENGE: Using only the assumption that angles in triangles sum to 180° - our fundamental postulate – prove Euclid's parallel postulate!

HINT: What goes wrong if there is "as much space as you could every want" between the two bold lines to the right? Draw a very very wide triangle.

At this point, a teacher can go two directions according to the textbook assigned to the geometry course:

IF YOUR GEOMETRY TEXT IS VERY TRADITIONAL: We have now motivated Euclid's Parallel Postulate and can make the agreement in class that we shall now replace our "flat postulate" with Euclid's postulate, noting that our flatness condition still holds as it follows from Euclid. One need not prove the tough challenge, but one does need to show how Euclid proved angles in a triangle sum to 180°.

IF YOU ARE NOT RESTRICTED TO A TRADITIONAL TEXT: You can stick with the flatness postulate to all of geometry, and note that you are not inconsistent with Euclid. Our flatness postulate implies his parallel postulate, and his parallel postulate implies our flatness postulate. They are logically the same start point. (So, actually, we've made traditional authors happy too!)

In case you are wondering …
In my geometry course we have just two more "big" fundamental postulates: The SAS postulate for triangles (motivated from the idea that we like to believe that straight lines possess the property "rise over run is constant" along the line) and Pythagoras's Theorem (motivated by the fact that Pythagoras's theorem is studied extensively in middle school and is assumed true there, so let's assume it is true here too!) This will shock most geometry authors, but it is true to the student experience of geometry!

[A wonderful surprise occurs when we discover later in the year that Pythagoras's Theorem logically follows from the SAS postulate. So our geometry actually boils down to just two significant postulates!]

The purists will argue that Euclid listed 10 postulates, and 2300 years later David Hilbert helped out Euclid by adding the 11 more he missed. Why don't I start by listing the 21 standard axioms? As I mentioned before, this is the **end** of the thinking process!

As a finale to my course I will ask: *What have we missed?* And that point we have had a year of playing with geometry (alas not years) and can begin to ask the question: *How should geometry begin?* just like Euclid did! Then, and only then, we might come up with 10 or so postulates about points and lines and their behaviours.

Oh .. If only we could design all curricula to be focused on process rather than just getting answers and test results!!

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